

Blow-up criteria of strong solutions to 3D compressible Navier-Stokes equations with vacuum

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Abstract

In the paper, we establish a blow-up criterion in terms of the integrability of the density for strong solutions to the Cauchy problem of compressible isentropic Navier-Stokes equations in \mathbb{R}^3 with vacuum, under the assumptions on the coefficients of viscosity: $\frac{29\mu}{3} > \lambda$. This extends the corresponding results in [20, 36] where a blow-up criterion in terms of the upper bound of the density was obtained under the condition $7\mu > \lambda$. As a byproduct, the restriction $7\mu > \lambda$ in [12, 37] is relaxed to $\frac{29\mu}{3} > \lambda$ for the full compressible Navier-Stokes equations by giving a new proof of Lemma 3.1. Besides, we get a blow-up criterion in terms of the upper bound of the density and the temperature for strong solutions to the Cauchy problem of the full compressible Navier-Stokes equations in \mathbb{R}^3 . The appearance of vacuum could be allowed. This extends the corresponding results in [37] where a blow-up criterion in terms of the upper bound of $(\rho, \frac{1}{\rho}, \theta)$ was obtained without vacuum. The effective viscous flux plays a very important role in the proofs.

Keyword: Compressible Navier-Stokes equations, strong solutions, blow-up criterion, vacuum.

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1 Introduction

The full compressible Navier-Stokes equations in \mathbb{R}^N are written as follows:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div}(\mathcal{T}), \\ (\rho E)_t + \operatorname{div}(\rho E u) + \operatorname{div}(P u) = \operatorname{div}(\mathcal{T} u) + \operatorname{div}(\kappa \nabla \theta). \end{cases} \quad (1.1)$$

Here \mathcal{T} is the stress tensor, given by

$$\mathcal{T} = \mu (\nabla u + (\nabla u)') + \lambda \operatorname{div} u I_N,$$

where I_N is a $N \times N$ unit matrix; $\rho = \rho(x, t)$, $u = u(x, t) : \mathbb{R}^N \times (0, \infty) \rightarrow \mathbb{R}^N$, and $\theta = \theta(x, t)$ are unknown functions denoting the density, velocity and absolute temperature, respectively; P , E and κ denote pressure, total energy and coefficient of heat conduction, respectively, where $E = e + \frac{|u|^2}{2}$ (e is the internal energy), and κ is a positive constant. Here, the state equations of P and e is of ideal polytropic gas type:

$$P = a\rho\theta, \quad e = C_0\theta,$$

where a and C_0 are two positive constants. μ and λ are the coefficients of viscosity, which are assumed to be constants, satisfying the following physical restrictions:

$$\mu > 0, \quad 2\mu + N\lambda \geq 0.$$

For isentropic fluids, the compressible Navier-Stokes equations become

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u. \end{cases} \quad (1.2)$$

Here P satisfies the equation of state of an ideal fluid:

$$P = a\rho^\gamma, \quad (a > 0, \quad \gamma > 1).$$

The compressible Navier-Stokes system is a well-known mathematical model which describes the motion of compressible fluids (refer for instance to [28] and references therein). There are so many known results on the well-posedness of solutions to (1.1) and (1.2). In the absence of vacuum (vacuum means $\rho = 0$), please refer for instance to [16, 17, 23, 25, 26, 27, 30, 31, 38] and references therein.

We give a brief survey on the well-posedness of solutions to (1.2) and (1.1) with vacuum. First, for (1.2), there has been made great progress since Lions' work. More precisely, the existence of global weak solutions to (1.2) with large initial data in \mathbb{R}^N was first obtained by Lions in [28], where $\gamma \geq \frac{3N}{N+2}$ for $N = 2$ or 3 . Feireisl et al in [15] extended Lions' work to the case $\gamma > \frac{3}{2}$ for $N = 3$. For solutions with spherical symmetry, Jiang and Zhang in [24] relaxed the restriction on γ in [28] to the case $\gamma > 1$, and got the global existence of the weak solutions for $N = 2$ or 3 . On the existence and regularity of weak solutions with density connecting to vacuum continuously in 1D, please refer to [29]. During the past two decades, Salvi, Choe, Kim and Jiang et al made progress towards the local or global existence of strong solutions with vacuum, see [4, 8, 11, 34]. On the classical solutions, refer to [6] for the local existence in three space dimension, and refer to [21] for global existence with small initial energy in 3D, and refer to [9] for global existence with large initial data in 1D. Secondly, for (1.1), the results on the global existence of weak solutions can be

referred to [2, 14]). More precisely, Feireisl in [14] got the global existence of *variational* solutions in dimension $N \geq 2$. The temperature equation in [14] is satisfied only as an inequality in the sense of distributions. Feireisl's work is the very first attempt towards the global existence of weak solutions to the full compressible Navier-Stokes equations in high dimensions. In order that the equations are satisfied as equalities in the sense of distribution, Bresch and Desjardins in [2] proposed some different assumptions from [14], and obtained the existence of global weak solutions to the full compressible Navier-Stokes equations with large initial data and density-dependent viscosities in \mathbb{T}^3 or \mathbb{R}^3 . On the regularities of the solutions to (1.1) when vacuum is allowed, please refer to [5] for the local existence and uniqueness of strong solutions in bounded or unbounded domains $\Omega \subseteq \mathbb{R}^3$, and refer to [40] for the global existence and uniqueness of classical solutions with large initial data in a bounded domain $I \subseteq \mathbb{R}^1$, and refer to [41] for the global existence and uniqueness of spherically or cylindrically symmetric classical solutions with large initial data in a bounded domain $\Omega \subseteq \mathbb{R}^3$.

It should be noted that one would not expect better regularities of the solutions of (1.1) or (1.2) in general because of Xin's results ([42]) and Rozanova's results ([33]). It was proved that there is no global smooth solution to the Cauchy problem of (1.1) or (1.2), if the initial density is nontrivial compactly supported ([42], $N = 1$ for (1.2) and $N \geq 1$ for (1.1)) or the solutions are highly decreasing at infinity ([33], $N \geq 3$ for (1.2) and (1.1)). In fact, a similar problem which is largely open for the incompressible Navier-Stokes equations in \mathbb{R}^3 , i.e., whether the global smooth solutions exist or not, was proposed as one of the Millennium Prize Problems by Clay Mathematics Institute (CMI) (see [3], 57-67: Charles L. Fefferman, Existence and Smoothness of the Navier-Stokes Equation). These motivate us to find some possible blow-up criterions of regular solutions to (1.1) and (1.2), especially of strong solutions. Such a problem has been studied for the incompressible Euler equations by Beale-Kato-Majda in their pioneering work [1], which showed that the $L_t^1 L_x^\infty$ -bound of vorticity $\nabla \times u$ alone controls the breakdown of smooth solutions. Later, Ponce [32] rephrased the BKM-criterion in terms of the deformation tensor $T_{ij} = \partial_j u^i + \partial_i u^j$. Recently, some results on the blow-up criterions have been done for some related models, such as compressible liquid crystal system which is the one coupling compressible Navier-Stokes equations with heat flow of harmonic map, see for instance [18, 19].

Before stating our main result, We would like to give some notations which will be used throughout the paper.

2 Main results

Before stating our main results, We would like to give some notations which will be used throughout the paper.

2.1 Notations

(i) $\int_{\mathbb{R}^3} f = \int_{\mathbb{R}^3} f \, dx.$

(ii) For $1 \leq l \leq \infty$, denote the L^l spaces and the standard Sobolev spaces as follows:

$$L^l = L^l(\mathbb{R}^3), \quad D^{k,l} = \left\{ u \in L_{\text{loc}}^1(\mathbb{R}^3) : \|\nabla^k u\|_{L^l} < \infty \right\},$$

$$W^{k,l} = L^l \cap D^{k,l}, \quad H^k = W^{k,2}, \quad D^k = D^{k,2},$$

$$D_0^1 = \left\{ u \in L^6 : \|\nabla u\|_{L^2} < \infty \right\},$$

$$\|u\|_{D^{k,l}} = \|\nabla^k u\|_{L^l}.$$

(iii) For two 3×3 matrices $E = (E_{ij}), F = (F_{ij})$, denote the scalar product between E and F by

$$E : F = \sum_{i,j=1}^3 E_{ij} F_{ij}.$$

(iv) $G = (2\mu + \lambda)\operatorname{div} u - P$ is the effective viscous flux.

(v) $\dot{h} = h_t + u \cdot \nabla h$ denotes the material derivative.

2.2 Compressible isentropic N-S: a blow-up criterion in terms of the integrability of the density

The constant a in the pressure function plays no roles in the analysis, we assume $a = 1$ henceforth. If the solutions are regular enough (such as strong solutions), (1.2) is equivalence to the following system which is very usefull in the proofs of the main theorems:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u + \nabla P = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u, \text{ in } \mathbb{R}^3. \end{cases} \quad (2.2.1)$$

System (2.2.1) is supplemented with initial conditions

$$(\rho, u)|_{t=0} = (\rho_0, u_0), \quad x \in \mathbb{R}^3, \quad (2.2.2)$$

with

$$\rho(x, t) \rightarrow 0, \quad u(x, t) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \quad t \geq 0. \quad (2.2.3)$$

We give the definition of strong solutions to (2.2.1) throughout the paper.

Definition 2.2.1 (*Strong solutions*) For $T > 0$, (ρ, u) is called a strong solution to the compressible Navier-Stokes equations (2.2.1)-(2.2.3) in $\mathbb{R}^3 \times [0, T]$, if for some $q \in (3, 6]$,

$$\begin{aligned} 0 \leq \rho &\in C([0, T]; W^{1,q} \cap H^1 \cap L^1), \quad \rho_t \in C([0, T]; L^2 \cap L^q), \\ u &\in C([0, T]; D^2 \cap D_0^1) \cap L^2(0, T; D^{2,q}), \quad u_t \in L^2(0, T; D_0^1), \quad \sqrt{\rho} u_t \in L^\infty(0, T; L^2), \end{aligned}$$

and (ρ, u) satisfies (2.2.1) a.e. in $\mathbb{R}^3 \times (0, T]$.

Our main result for compressible isentropic Navier-Stokes equations is stated as follows:

Theorem 2.2.2 Assume $\rho_0 \geq 0$, $\rho_0 \in L^1 \cap H^1 \cap W^{1,q}$, for some $q \in (3, 6]$, $u_0 \in D^2 \cap D_0^1$, and the following compatibility conditions are satisfied:

$$\mu \Delta u_0 + (\mu + \lambda) \nabla \operatorname{div} u_0 - \nabla P(\rho_0) = \sqrt{\rho_0} g, \quad x \in \mathbb{R}^3, \quad (2.2.4)$$

for some $g \in L^2$. Let (ρ, u) be a strong solution to (2.2.1)-(2.2.3) in $\mathbb{R}^3 \times [0, T]$. If $0 < T^* < +\infty$ is the maximum time of existence of the strong solution, then

$$\limsup_{T \nearrow T^*} \|\rho\|_{L^\infty(0, T; L^{q_1})} = \infty, \quad (2.2.5)$$

for some $1 < q_1 < \infty$ large enough, provided $\frac{29\mu}{3} > \lambda$.

Remark 2.2.3 *Under the conditions of Theorem 2.2.2, the local existence of the strong solutions was obtained in [4]. Thus, the assumption $T^* > 0$ makes sense.*

In the presence of vacuum, before Theorem 2.2.2, there are several results on the blow-up criterions of strong solutions to (2.2.1), refer for instance to [4, 10, 20, 22, 36]. More precisely, let $0 < T^* < +\infty$ is the maximum time of existence of strong solutions. Then the blow-up criterions can be summed as follows:

- Cho-Choe-Kim ([4])

$$\limsup_{t \nearrow T^*} (\|\rho(t)\|_{H^1 \cap W^{1,q}} + \|u(t)\|_{D_0^1}) = \infty, \quad (2.2.6)$$

for some $q \in (3, 6]$;

- Fan-Jiang ([10])

$$\limsup_{t \nearrow T^*} \left(\|\rho(t)\|_{L^\infty} + \int_0^t (\|\rho(s)\|_{W^{1,q}} + \|\nabla \rho(s)\|_{L^2}^4) ds \right) = \infty, \quad (2.2.7)$$

for some $q \in (3, 6]$, provided $7\mu > 9\lambda$;

- Huang-Li-Xin ([22])

$$\limsup_{t \nearrow T^*} \int_0^t \left\| \frac{\nabla u(s) + (\nabla u)'(s)}{2} \right\|_{L^\infty} ds = \infty; \quad (2.2.8)$$

- Huang-Xin ([20]) (Serrin's criterion [35])

$$\limsup_{t \nearrow T^*} (\|\rho\|_{L^\infty(0,t;L^\infty)} + \|\sqrt{\rho}u\|_{L^s(0,t;L^r)}) = \infty, \quad (2.2.9)$$

where $\frac{2}{s} + \frac{3}{r} \leq 1$, $3 < r \leq \infty$;

- Huang-Xin ([20], for Cauchy problem), Sun-Wang-Zhang ([36], for Cauchy problem and IBVP)

$$\limsup_{t \nearrow T^*} \|\rho\|_{L^\infty(0,t;L^\infty)} = \infty, \quad (2.2.10)$$

provided $7\mu > \lambda$.

We introduce the main ideas of the proof of Theorem 2.2.2, some of which are inspired by some of the arguments in [7, 20, 36, 39].

(1) In [20, 36], to prove (2.2.10), the restriction $7\mu > \lambda$ plays an important role in the analysis. In fact, the condition $7\mu > \lambda$ is only used to get the upper bound of $\int_{\mathbb{R}^3} \rho|u|^r$, for some $r > 3$, so is it for (2.3.6) and (2.3.8). Here, we get the upper bound of $\int_{\mathbb{R}^3} \rho|u|^r$, under the assumption $\frac{29\mu}{3} > \lambda$ (see Lemma 3.1), which as a byproduct of Lemma 3.1 extends the results in [12, 20, 36, 37] (see Remark 3.2). From the proof of Lemma 3.1, we know that it is important to handle the second term of the right hand side of (3.4) where $\operatorname{div} u$ and $|\nabla u|$ are involved. On the other hand, the second term of the left hand side of (3.4), where $|\nabla u|^2$, $|\operatorname{div} u|^2$ and $|\nabla|u||^2$ are involved, is not enough to absorb the second term of the right under the physical restrictions of the viscosities. For the term $|\nabla u|^2$ on the left of (3.4), it is natural to get $|\nabla u|^2 \geq |\nabla|u||^2$, which makes some additional good information on $|\nabla u|^2$ lose cf. [20, 36]. The crucial ingredient to relax the additional restrictions to $\frac{29\mu}{3} > \lambda$ is that we observe

$$|\nabla u|^2 = |u|^2 \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 + |\nabla|u||^2,$$

for $|u| > 0$, and thus

$$\int_{\mathbb{R}^3 \cap \{|u| > 0\}} |u|^{r-2} |\nabla u|^2 \geq (1 + \phi(\varepsilon_1, r)) \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |u|^{r-2} |\nabla |u||^2,$$

if

$$\int_{\mathbb{R}^3 \cap \{|u| > 0\}} |u|^r \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 \geq \phi(\varepsilon_1, r) \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |u|^{r-2} |\nabla |u||^2,$$

for some positive function $\phi(\varepsilon_1, r)$ near $r = 3$. For more details, please see Lemma 3.1.

(2) In [7], the authors obtain the upper bound and the positive lower bound of the density in \mathbb{T}^3 under the assumptions $\inf \rho_0 > 0$, $\mu + \lambda = 0$ and $\|\rho\|_{L^\infty(0, T; L^{q_0})}$ is bounded, for some $q_0 > 0$ large enough and for some $T > 0$. From the physical points of view, $\mu + \lambda > 0$ seems more natural, since we know that $\mu > 0$ and $2\mu + 3\lambda \geq 0$ deduce $\mu + \lambda > 0$. In Theorem 2.2.2, we only assume $\mu > 0$, $2\mu + 3\lambda \geq 0$ and $\frac{29\mu}{3} > \lambda$.

(3) By (2.2.1)₁, we know $\|\rho\|_{L^1} = \|\rho_0\|_{L^1}$. It follows from the standard interpolation inequality that the bound of $\|\rho\|_{L^\infty(0, T; L^\infty)}$ yields that $\|\rho\|_{L^\infty(0, T; L^{q_2})}$ is bounded for any $q_2 \in (1, \infty)$. Thus, the blow-up criterion (2.2.5) is an extension towards (2.2.10) in [20, 36].

2.3 Full compressible N-S: a blow-up criterion in terms of the upper bound of the density and the temperature

The constants a , C_0 and κ in the equations play no roles in the analysis, we assume $a = C_0 = \kappa = 1$ henceforth. If the solutions are regular enough (such as strong solutions), (1.1) is equivalence to the following system which is very usefull in the proofs of the main theorems:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u + \nabla P(\rho, \theta) = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u, \\ \rho \theta_t + \rho u \cdot \nabla \theta + \rho \theta \operatorname{div} u = \frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\operatorname{div} u)^2 + \Delta \theta, \text{ in } \mathbb{R}^3. \end{cases} \quad (2.3.1)$$

System (2.3.1) is supplemented with initial conditions

$$(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0), \quad x \in \mathbb{R}^3, \quad (2.3.2)$$

with

$$\rho(x, t) \rightarrow 0, \quad u(x, t) \rightarrow 0, \quad \theta(x, t) \rightarrow 0, \quad \text{as } |x| \rightarrow \infty, \quad \text{for } t \geq 0. \quad (2.3.3)$$

We give the definition of strong solutions to (2.3.1) throughout the paper.

Definition 2.3.1 (*Strong solution*) For $T > 0$, (ρ, u, θ) is called a strong solution to the compressible Navier-Stokes equations (2.3.1)-(2.3.3) in $\mathbb{R}^3 \times [0, T]$, if for some $q \in (3, 6]$,

$$\begin{aligned} 0 \leq \rho &\in C([0, T]; W^{1, q} \cap H^1 \cap L^1), \quad \rho_t \in C([0, T]; L^2 \cap L^q), \\ (u, \theta) &\in C([0, T]; D^2 \cap D_0^1) \cap L^2(0, T; D^{2, q}), \quad (u_t, \theta_t) \in L^2(0, T; D_0^1), \\ (\sqrt{\rho} u_t, \sqrt{\rho} \theta_t) &\in L^\infty(0, T; L^2), \end{aligned}$$

and (ρ, u, θ) satisfies (2.3.1) a.e. in $\mathbb{R}^3 \times (0, T]$.

Our main result for the full compressible Navier-Stokes equations is stated as follows:

Theorem 2.3.2 Assume $\rho_0 \geq 0$, $\rho_0 \in H^1 \cap W^{1,q} \cap L^1$, for some $q \in (3, 6]$, $(u_0, \theta_0) \in D^2 \cap D_0^1$, and the following compatibility conditions are satisfied:

$$\begin{cases} \mu \Delta u_0 + (\mu + \lambda) \nabla \operatorname{div} u_0 - \nabla P(\rho_0, \theta_0) = \sqrt{\rho_0} g_1, \\ \kappa \Delta \theta_0 + \frac{\mu}{2} |\nabla u_0 + (\nabla u_0)'|^2 + \lambda (\operatorname{div} u_0)^2 = \sqrt{\rho_0} g_2, \quad x \in \mathbb{R}^3, \end{cases} \quad (2.3.4)$$

for some $g_i \in L^2$, $i = 1, 2$. Let (ρ, u, θ) be a strong solution to (2.3.1)-(2.3.3) in $\mathbb{R}^3 \times [0, T]$. If $0 < T^* < +\infty$ is the maximum time of existence of the strong solution, then

$$\limsup_{T \nearrow T^*} (\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\theta\|_{L^\infty(0,T;L^\infty)}) = \infty, \quad (2.3.5)$$

provided $3\mu > \lambda$.

Remark 2.3.3 Under the conditions of Theorem 2.3.2, the local existence of the strong solutions was obtained in [5]. Thus, the assumption $T^* > 0$ makes sense.

Remark 2.3.4 Theorem 2.3.2 is also valid for more general pressure law, such as $P = a\rho\theta + a_1\rho^\gamma$. Whether the similar result as in Theorem 2.2.2 could be obtained for the full compressible Navier-Stokes equations is still unknown.

Before Theorem 2.3.2, there are several results on the blow-up criterions of strong solutions to (2.3.1), please refer for instance to [12, 13, 37] and references therein for initial boundary value problems. In particular,

- Fan-Jiang-Ou ([12], 3D)

$$\limsup_{t \nearrow T^*} (\|\theta\|_{L^\infty(0,t;L^\infty)} + \|\nabla u\|_{L^1(0,t;L^\infty)}) = \infty, \quad (2.3.6)$$

provided $7\mu > \lambda$. Here the appearance of vacuum is allowed.

It is well-known that the bound of $\|\nabla u\|_{L^1(0,t;L^\infty)}$ yields that $\|\rho\|_{L^\infty(0,t;L^\infty)}$ is bounded (see (2.2) in [12]), if the initial density is bounded. When $\|\nabla u\|_{L^1(0,t;L^\infty)}$ in (2.3.6) is replaced by the upper bound of the density, the following blow-up criterions were obtained:

- Fang-Zi-Zhang ([13], 2D)

$$\limsup_{t \nearrow T^*} (\|\theta\|_{L^\infty(0,t;L^\infty)} + \|\rho\|_{L^\infty(0,t;L^\infty)}) = \infty, \quad (2.3.7)$$

where the appearance of vacuum is allowed;

- Sun-Wang-Zhang ([37], 3D)

$$\limsup_{t \nearrow T^*} \left(\|\theta\|_{L^\infty(0,t;L^\infty)} + \|\rho\|_{L^\infty(0,t;L^\infty)} + \left\| \frac{1}{\rho} \right\|_{L^\infty(0,t;L^\infty)} \right) = \infty, \quad (2.3.8)$$

provided $7\mu > \lambda$.

We would like to point out that an analogous blow-up criterion of (2.3.5) for the isentropic compressible Navier-Stokes equation (i.e. (2.2.10)) in \mathbb{R}^3 , under the assumption $7\mu > \lambda$, has been previously established by Huang-Li-Xin [20] and Sun-Wang-Zhang [36]. In [20, 36], the restriction $7\mu > \lambda$ was needed only for the estimate of $\int \rho|u|^{3+\delta}$ where $\delta > 0$ is sufficiently small.

We introduce the main ideas of the proof of Theorem 2.3.2.

(1) To get the upper bound of $\int_{\mathbb{R}^3} \rho|u|^r$, we apply the ideas of the proof of Lemma 3.1 so that we can get a restriction of μ and λ as better as possible. As a byproduct, we also get the upper

bound of $\int_0^t \int_{\mathbb{R}^3} |u|^{r-2} |\nabla u|^2$, which is very crucial in the proof of $L_t^\infty L_x^2$ of ∇u (see Lemma 4.3). Here we take $r = 4$ because we have to deal with the difficulties caused by the strong nonlinearities in the temperature equation, such as the terms $\frac{\mu}{2} |\nabla u + (\nabla u)'|^2$ and $\lambda(\operatorname{div} u)^2$ in (2.3.1)₃, which leads to the restriction $3\mu > \lambda$.

(2) As it was pointed out in [37] that to deal with the essential difficulties due to the highly nonlinear terms $|\nabla u + (\nabla u)'|^2$ and $|\operatorname{div} u|^2$ in the temperature equation, Sun-Wang-Zhang used the ideas of Hoff [17] to get the upper bounds of $L_t^\infty H_x^s$ of u for $s \in (0, 1)$, which requires the upper bound of $\frac{1}{\rho}$. Here we do not require the upper bound of $\frac{1}{\rho}$ so that the appearance of vacuum is allowed, because we use the fact $P_t = (\rho E)_t - \left(\frac{\rho|u|^2}{2}\right)_t$, (1.1)₃ and integration by parts such that

$$\begin{aligned} - \int_{\mathbb{R}^3} P_t G &= - \int_{\mathbb{R}^3} (\rho E)_t G + \dots = - \int_{\mathbb{R}^3} \operatorname{div} \left[\left(\mu (\nabla u + (\nabla u)') + \lambda \operatorname{div} u I_N \right) u \right] G + \dots \\ &= \int_{\mathbb{R}^3} \left[\left(\mu (\nabla u + (\nabla u)') + \lambda \operatorname{div} u I_N \right) u \right] \cdot \nabla G + \dots \\ &\leq C \| |u| |\nabla u| \|_{L^2} \| \nabla G \|_{L^2} + \dots, \end{aligned}$$

where $G = (2\mu + \lambda)\operatorname{div} u - P$ is the effective viscous flux which plays an important role in the proofs. For more details, please see (4.19)-(4.32) in the proof of Lemma 4.3.

(3) The nonlinear terms $|\nabla u + (\nabla u)'|^2$ and $|\operatorname{div} u|^2$ in (2.3.1)₃ could be handled for two space dimension when the blow-up criterion (2.3.7) was established with vacuum, because 2-D Gagliardo-Nirenberg inequality has better properties than 3-D. See [13] for more details.

3 Proof of Theorem 2.2.2

Let $0 < T^* < \infty$ be the maximum time of existence of strong solution (ρ, u) to (2.2.1)-(2.2.3). Namely, (ρ, u) is a strong solution to (2.2.1)-(2.2.3) in $\mathbb{R}^3 \times [0, T]$ for any $0 < T < T^*$, but not a strong solution in $\mathbb{R}^3 \times [0, T^*]$. Suppose that (2.2.5) were false, i.e.

$$M := \|\rho\|_{L^\infty(0, T^*; L^{q_1})} < \infty. \quad (3.1)$$

The goal is to show that under the assumption (3.1), there is a bound $C > 0$ depending only on $M, \rho_0, u_0, \mu, \lambda$, and T^* such that

$$\sup_{0 \leq t < T^*} \left[\max_{l=2, q} (\|\rho\|_{W^{1, l}} + \|\rho_t\|_{L^l}) + \|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u\|_{H^1} \right] \leq C, \quad (3.2)$$

and

$$\int_0^{T^*} (\|u_t\|_{D^1}^2 + \|u\|_{D^{2, q}}^2) dt \leq C. \quad (3.3)$$

With (3.2) and (3.3), it is easy to show without much difficulties that T^* is not the maximum time, which is the desired contradiction.

Throughout the rest of the section, we denote by C a generic constant depending only on $\rho_0, u_0, T^*, M, \lambda, \mu$. We denote by

$$A \lesssim B$$

if there exists a generic constant C such that $A \leq CB$.

Lemma 3.1 *Under the conditions of Theorem 2.2.2 and (3.1), if $\frac{29\mu}{3} > \lambda$, there exists $r \in (3, \frac{7}{2})$ such that*

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \rho |u|^r dx \leq C,$$

for any $T \in [0, T^*)$.

Remark 3.2 *Lemma 3.1 is also true for bounded domains. This lemma relaxes the restriction $7\mu > \lambda$ in [20, 36] to $\frac{29\mu}{3} > \lambda$. It is easy to verify that Lemma 3.1 is also true if $P = R\rho\theta$ for a constant $R > 0$ and θ is bounded. Thus, as a byproduct of the paper, the restriction $7\mu > \lambda$ in [12, 37] could be relaxed to $\frac{29\mu}{3} > \lambda$ for the full compressible Navier-Stokes equations. In this sense, this lemma extends the results in [12, 20, 36, 37].*

Proof. Multiplying (2.2.1)₂ by $r|u|^{r-2}u$, and integrating by parts over \mathbb{R}^3 , we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r + \int_{\mathbb{R}^3} r |u|^{r-2} (\mu |\nabla u|^2 + (\lambda + \mu) |\operatorname{div} u|^2 + \mu(r-2) |\nabla |u||^2) \\ &= r \int_{\mathbb{R}^3} \operatorname{div}(|u|^{r-2}u)P - r(r-2)(\mu + \lambda) \int_{\mathbb{R}^3} \operatorname{div} u |u|^{r-3} u \cdot \nabla |u|. \end{aligned}$$

Thus,

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r + \int_{\mathbb{R}^3 \cap \{|u|>0\}} r |u|^{r-2} (\mu |\nabla u|^2 + (\lambda + \mu) |\operatorname{div} u|^2 + \mu(r-2) |\nabla |u||^2) \\ &= r \int_{\mathbb{R}^3 \cap \{|u|>0\}} \operatorname{div}(|u|^{r-2}u)P - r(r-2)(\mu + \lambda) \int_{\mathbb{R}^3 \cap \{|u|>0\}} \operatorname{div} u |u|^{r-3} u \cdot \nabla |u|. \end{aligned} \quad (3.4)$$

For any given $\varepsilon_1 \in (0, 1)$, we define a nonnegative function which will be decided in **Case 2** as follows:

$$\phi(\varepsilon_1, r) = \begin{cases} \frac{\mu\varepsilon_1(r-1)}{3\left(-\frac{4\mu}{3}-\lambda+\frac{r^2(\mu+\lambda)}{4(r-1)}\right)}, & \text{if } \frac{r^2(\mu+\lambda)}{4(r-1)} - \frac{4\mu}{3} - \lambda > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Case 1:

$$\int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^r \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 > \phi(\varepsilon_1, r) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2. \quad (3.5)$$

A direct calculation gives for $|u| > 0$

$$|\nabla u|^2 = |u|^2 \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 + |\nabla |u||^2, \quad (3.6)$$

which plays a important role in the proof.

By (3.4), we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r + \int_{\mathbb{R}^3 \cap \{|u|>0\}} r |u|^{r-2} (\mu |\nabla u|^2 + (\lambda + \mu) |\operatorname{div} u|^2 + \mu(r-2) |\nabla |u||^2) \\ &= r \int_{\mathbb{R}^3 \cap \{|u|>0\}} \operatorname{div}(|u|^{r-2}u)P - r(r-2)(\mu + \lambda) \int_{\mathbb{R}^3 \cap \{|u|>0\}} \operatorname{div} u |u|^{\frac{r-2}{2}} |u|^{\frac{r-4}{2}} u \cdot \nabla |u| \\ &\leq C \int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho^{\gamma-\frac{r-2}{2r}} \rho^{\frac{r-2}{2r}} |u|^{r-2} |\nabla u| + r(\mu + \lambda) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\operatorname{div} u|^2 \\ &\quad + \frac{r(r-2)^2(\mu + \lambda)}{4} \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2, \end{aligned}$$

where we have used Cauchy inequality. Thus,

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r + \int_{\mathbb{R}^3 \cap \{|u|>0\}} \mu r |u|^{r-2} |\nabla u|^2 + \mu(r-2)r \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 \\ & \leq C \int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho^{\gamma-\frac{r-2}{2r}} \rho^{\frac{r-2}{2r}} |u|^{r-2} |\nabla u| + \frac{r(r-2)^2(\mu+\lambda)}{4} \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2. \end{aligned} \quad (3.7)$$

By (3.6), (3.7), Cauchy inequality, and Hölder inequality, for any $\varepsilon_0 \in (0, 1)$, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r + \int_{\mathbb{R}^3 \cap \{|u|>0\}} \mu r |u|^{r-2} |\nabla u|^2 + \int_{\mathbb{R}^3 \cap \{|u|>0\}} \mu r |u|^r \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 \\ & + \mu(r-2)r \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 \\ & \leq C \int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho^{\gamma-\frac{r-2}{2r}} \rho^{\frac{r-2}{2r}} |u|^{r-2} |\nabla u| + C \int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho^{\gamma-\frac{r-2}{2r}} \rho^{\frac{r-2}{2r}} |u|^{r-1} \left| \nabla \left(\frac{u}{|u|} \right) \right| \\ & + \frac{r(r-2)^2(\mu+\lambda)}{4} \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 \\ & \leq C \int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho^{\gamma-\frac{r-2}{2r}} \rho^{\frac{r-2}{2r}} |u|^{r-2} |\nabla u| + \mu r \varepsilon_0 \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^r \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 \\ & + \frac{C}{4\mu r \varepsilon_0} \left(\int_{\mathbb{R}^3} \rho |u|^r \right)^{\frac{r-2}{r}} \left(\int_{\mathbb{R}^3} \rho^{\frac{(2\gamma-1)r}{2}+1} \right)^{\frac{2}{r}} + \frac{r(r-2)^2(\mu+\lambda)}{4} \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2. \end{aligned}$$

Combining (3.1) and (3.5), we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r + r \left[\mu(1-\varepsilon_0)\phi(\varepsilon_1, r) + \mu(r-1) - \frac{(r-2)^2(\mu+\lambda)}{4} \right] \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 \\ & \leq C \int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho^{\gamma-\frac{r-2}{2r}} \rho^{\frac{r-2}{2r}} |u|^{r-2} |\nabla u| + \frac{C}{4\mu r \varepsilon_0} \left(\int_{\mathbb{R}^3} \rho |u|^r \right)^{\frac{r-2}{r}}. \end{aligned} \quad (3.8)$$

(Sub-Case 1₁): If $3 \in \{r|\frac{r^2(\mu+\lambda)}{4(r-1)} - \frac{4\mu}{3} - \lambda > 0\}$, i.e., $5\mu < 3\lambda$, it is easy to get $[3, \infty) \subset \{r|\frac{r^2(\mu+\lambda)}{4(r-1)} - \frac{4\mu}{3} - \lambda > 0\}$. Therefore, we have

$$\phi(\varepsilon_1, r) = \frac{\mu \varepsilon_1 (r-1)}{3 \left(-\frac{4\mu}{3} - \lambda + \frac{r^2(\mu+\lambda)}{4(r-1)} \right)}, \quad (3.9)$$

for any $r \in [3, \infty)$.

Denote

$$f(\varepsilon_0, \varepsilon_1, r) = \mu(1-\varepsilon_0)\phi(\varepsilon_1, r) + \mu(r-1) - \frac{(r-2)^2(\mu+\lambda)}{4}. \quad (3.10)$$

Substituting (3.9) into (3.10), for $r \in [3, \infty)$, we have

$$f(\varepsilon_0, \varepsilon_1, r) = \frac{\mu^2 \varepsilon_1 (1-\varepsilon_0)(r-1)}{3 \left(-\frac{4\mu}{3} - \lambda + \frac{r^2(\mu+\lambda)}{4(r-1)} \right)} + \mu(r-1) - \frac{(r-2)^2(\mu+\lambda)}{4}. \quad (3.11)$$

For $(\varepsilon_0, \varepsilon_1, r) = (0, 1, 3)$, we have

$$f(0, 1, 3) = \frac{16\mu^2}{3\lambda - 5\mu} + \frac{7\mu - \lambda}{4} > 0,$$

where we have used $\frac{5\mu}{3} < \lambda < \frac{29}{3}\mu$.

Since $f(\varepsilon_0, \varepsilon_1, r)$ is continuous w.r.t. $(\varepsilon_0, \varepsilon_1, r)$ over $[0, 1] \times [0, 1] \times [3, \infty)$, there exist $\varepsilon_0, \varepsilon_1 \in (0, 1)$ and $r \in (3, \frac{7}{2})$, such that

$$f(\varepsilon_0, \varepsilon_1, r) > 0.$$

By (3.8), Cauchy inequality and Hölder inequality, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r + r f(\varepsilon_0, \varepsilon_1, r) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 \\ & \leq r f(\varepsilon_0, \varepsilon_1, r) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 + \frac{C}{4r f(\varepsilon_0, \varepsilon_1, r)} \left(\int_{\mathbb{R}^3} \rho |u|^r \right)^{\frac{r-2}{r}} \left(\int_{\mathbb{R}^3} \rho^{\frac{(2\gamma-1)r}{2}+1} \right)^{\frac{2}{r}} \\ & \quad + \frac{C}{4\mu r \varepsilon_0} \left(\int_{\mathbb{R}^3} \rho |u|^r \right)^{\frac{r-2}{r}}. \end{aligned}$$

This together with (3.1) gives

$$\frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r \leq C \left[\frac{1}{f(\varepsilon_0, \varepsilon_1, r)} + \frac{1}{\mu \varepsilon_0} \right] \left(\int_{\mathbb{R}^3} \rho |u|^r \right)^{\frac{r-2}{r}}. \quad (3.12)$$

(Sub-Case 1₂): if $3 \notin \{r | \frac{r^2(\mu+\lambda)}{4(r-1)} - \frac{4\mu}{3} - \lambda > 0\}$, i.e., $5\mu \geq 3\lambda$.

In this case, for $r \in (3, \frac{7}{2})$, it is easy to get

$$\begin{aligned} & r \left[\mu(1 - \varepsilon_0) \phi(\varepsilon_1, r) + \mu(r-1) - \frac{(r-2)^2(\mu+\lambda)}{4} \right] \\ & > 3 \left(2\mu - \frac{9(\mu+\lambda)}{16} \right) = 3 \left(\frac{23\mu}{16} - \frac{9\lambda}{16} \right) \\ & \geq 3 \left(\frac{23\mu}{16} - \frac{15\mu}{16} \right) = \frac{3\mu}{2}. \end{aligned} \quad (3.13)$$

By (3.8), (3.13), Cauchy inequality and Hölder inequality, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r + \frac{3\mu}{2} \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 \\ & \leq C \int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho^{\gamma-\frac{r-2}{2r}} \rho^{\frac{r-2}{2r}} |u|^{r-2} |\nabla |u|| + \frac{C}{4\mu r \varepsilon_0} \left(\int_{\mathbb{R}^3} \rho |u|^r \right)^{\frac{r-2}{r}} \\ & \leq \frac{3\mu}{2} \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 + C \left(\int_{\mathbb{R}^3} \rho |u|^r \right)^{\frac{r-2}{r}} \left(\int_{\mathbb{R}^3} \rho^{\frac{(2\gamma-1)r}{2}+1} \right)^{\frac{2}{r}} + \frac{C}{4\mu r \varepsilon_0} \left(\int_{\mathbb{R}^3} \rho |u|^r \right)^{\frac{r-2}{r}}. \end{aligned}$$

Therefore,

$$\frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r \leq C \left(\int_{\mathbb{R}^3} \rho |u|^r \right)^{\frac{r-2}{r}}, \quad (3.14)$$

where we have used (3.1).

By (3.12) and (3.14), for **Case 1**, we conclude that if $\lambda < \frac{29}{3}\mu$ and (3.5) are satisfied, the following estimate can be obtained

$$\frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r \leq C \left(\int_{\mathbb{R}^3} \rho |u|^r \right)^{\frac{r-2}{r}}, \quad (3.15)$$

for some constants $C > 0$ and $r \in (3, \frac{7}{2})$.

Case 2: if

$$\int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^r \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 \leq \phi(\varepsilon_1, r) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2. \quad (3.16)$$

A direct calculation gives for $|u| > 0$

$$\operatorname{div} u = |u| \operatorname{div} \left(\frac{u}{|u|} \right) + \frac{u \cdot \nabla |u|}{|u|}. \quad (3.17)$$

By (3.4) and (3.17), we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r + \int_{\mathbb{R}^3 \cap \{|u|>0\}} r |u|^{r-2} \left(\mu |\nabla u|^2 + (\lambda + \mu) |\operatorname{div} u|^2 + \mu(r-2) |\nabla |u||^2 \right) \\ &= r \int_{\mathbb{R}^3 \cap \{|u|>0\}} \operatorname{div}(|u|^{r-2} u) P - r(r-2)(\mu + \lambda) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} u \cdot \nabla |u| \operatorname{div} \left(\frac{u}{|u|} \right) \\ & \quad - r(r-2)(\mu + \lambda) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-4} |u \cdot \nabla |u||^2. \end{aligned}$$

This gives

$$\frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r + \int_{\mathbb{R}^3 \cap \{|u|>0\}} r |u|^{r-4} G = r \int_{\mathbb{R}^3 \cap \{|u|>0\}} \operatorname{div}(|u|^{r-2} u) P, \quad (3.18)$$

where

$$\begin{aligned} G &= \mu |u|^2 |\nabla u|^2 + (\lambda + \mu) |u|^2 |\operatorname{div} u|^2 + \mu(r-2) |u|^2 |\nabla |u||^2 \\ & \quad + (r-2)(\mu + \lambda) |u|^2 u \cdot \nabla |u| \operatorname{div} \left(\frac{u}{|u|} \right) + (r-2)(\mu + \lambda) |u \cdot \nabla |u||^2. \end{aligned}$$

To let $\int_{\mathbb{R}^3 \cap \{|u|>0\}} r |u|^{r-4} G$ become a *good* term, we shall consider G first.

$$\begin{aligned} G &= \mu |u|^2 \left(|u|^2 \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 + |\nabla |u||^2 \right) + (\mu + \lambda) |u|^2 \left(|u| \operatorname{div} \left(\frac{u}{|u|} \right) + \frac{u \cdot \nabla |u|}{|u|} \right)^2 \\ & \quad + \mu(r-2) |u|^2 |\nabla |u||^2 + (r-2)(\mu + \lambda) |u|^2 u \cdot \nabla |u| \operatorname{div} \left(\frac{u}{|u|} \right) + (r-2)(\mu + \lambda) |u \cdot \nabla |u||^2 \\ &= \mu |u|^4 \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 + \mu(r-1) |u|^2 |\nabla |u||^2 + (r-1)(\mu + \lambda) |u \cdot \nabla |u||^2 \\ & \quad + r(\mu + \lambda) |u|^2 u \cdot \nabla |u| \operatorname{div} \left(\frac{u}{|u|} \right) + (\mu + \lambda) |u|^4 \left(\operatorname{div} \left(\frac{u}{|u|} \right) \right)^2 \\ &= \mu |u|^4 \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 + \mu(r-1) |u|^2 |\nabla |u||^2 + (r-1)(\mu + \lambda) \left(u \cdot \nabla |u| + \frac{r}{2(r-1)} |u|^2 \operatorname{div} \left(\frac{u}{|u|} \right) \right)^2 \\ & \quad + (\mu + \lambda) |u|^4 \left(\operatorname{div} \left(\frac{u}{|u|} \right) \right)^2 - \frac{r^2(\mu + \lambda)}{4(r-1)} |u|^4 \left(\operatorname{div} \left(\frac{u}{|u|} \right) \right)^2. \end{aligned}$$

This, combining the fact

$$\left| \operatorname{div} \left(\frac{u}{|u|} \right) \right|^2 \leq 3 \left| \nabla \left(\frac{u}{|u|} \right) \right|^2,$$

deduces

$$\begin{aligned} G &\geq \mu |u|^4 \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 + \mu(r-1) |u|^2 |\nabla |u||^2 + \left(\mu + \lambda - \frac{r^2(\mu + \lambda)}{4(r-1)} \right) |u|^4 \left(\operatorname{div} \left(\frac{u}{|u|} \right) \right)^2 \\ &\geq \frac{\mu}{3} |u|^4 \left(\operatorname{div} \left(\frac{u}{|u|} \right) \right)^2 + \left(\mu + \lambda - \frac{r^2(\mu + \lambda)}{4(r-1)} \right) |u|^4 \left(\operatorname{div} \left(\frac{u}{|u|} \right) \right)^2 + \mu(r-1) |u|^2 |\nabla |u||^2 \\ &= \left(\frac{4\mu}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r-1)} \right) |u|^4 \left(\operatorname{div} \left(\frac{u}{|u|} \right) \right)^2 + \mu(r-1) |u|^2 |\nabla |u||^2. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^3 \cap \{|u|>0\}} r |u|^{r-4} G &\geq r \left(\frac{4\mu}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r-1)} \right) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^r \left(\operatorname{div} \left(\frac{u}{|u|} \right) \right)^2 \\ &\quad + \mu r(r-1) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 \\ &\geq 3r \left(\frac{4\mu}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r-1)} \right) \phi(\varepsilon_1, r) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 \\ &\quad + \mu r(r-1) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 \\ &= \left[3r \left(\frac{4\mu}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r-1)} \right) \phi(\varepsilon_1, r) + \mu r(r-1) \right] \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2, \end{aligned}$$

where we have used (3.16).

Putting all these estimates into (3.18), we have

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r + \left[3r \left(\frac{4\mu}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r-1)} \right) \phi(\varepsilon_1, r) + \mu r(r-1) \right] \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 \\ &\leq C \int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho^{\gamma - \frac{r-2}{2r}} \rho^{\frac{r-2}{2r}} |u|^{r-2} |\nabla |u||^2 \\ &\leq \varepsilon \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 + \frac{C}{\varepsilon} \left(\int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho |u|^r \right)^{\frac{r-2}{r}} \left(\int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho^{\frac{(2\gamma-1)r}{2} + 1} \right)^{\frac{2}{r}} \\ &\leq \varepsilon (1 + \phi(\varepsilon_1, r)) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 + \frac{C}{\varepsilon} \left(\int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho |u|^r \right)^{\frac{r-2}{r}}, \end{aligned}$$

where we have used Cauchy inequality, Hölder inequality and (3.1).

Taking $\varepsilon = (1 + \phi(\varepsilon_1, r))^{-1} \left[3r \left(\frac{4\mu}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r-1)} \right) \phi(\varepsilon_1, r) + \mu r(r-1) \right]$, we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r \leq \frac{C(1 + \phi(\varepsilon_1, r))}{\left[3r \left(\frac{4\mu}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r-1)} \right) \phi(\varepsilon_1, r) + \mu r(r-1) \right]} \left(\int_{\mathbb{R}^3} \rho |u|^r \right)^{\frac{r-2}{r}}, \quad (3.19)$$

for $r \in (3, \frac{7}{2})$.

By (3.15) and (3.19), for **Case 1** and **Case 2**, we conclude that if $\lambda < \frac{29}{3}\mu$, there exist some constants $C > 0$ and $r \in (3, \frac{7}{2})$ such that

$$\frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r \leq C \left(\int_{\mathbb{R}^3} \rho |u|^r \right)^{\frac{r-2}{r}}. \quad (3.20)$$

Since $\frac{r-2}{r} \in (0, 1)$, using Young inequality and Gronwall inequality over (3.20), we complete the proof of Lemma 3.1. \square

From Remark 3.2 and [20, 36], in order to get (3.2) and (3.3), it suffices to get the upper bound of $\sup_{0 \leq t < T^*} \|\rho(t)\|_{L^\infty}$. To do this, Lemma 3.3 and Lemma 3.4 are needed.

Lemma 3.3 *Under the conditions of Theorem 2.2.2 and (3.1), it holds that for any $T \in [0, T^*)$*

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^T \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \leq C,$$

where $\dot{u} = u_t + u \cdot \nabla u$ by the definition of the material derivative.

Proof. Multiplying (2.2.1)₂ by u_t , and integrating by parts over \mathbb{R}^3 , we have

$$\begin{aligned} & \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + (\mu + \lambda) |\operatorname{div} u|^2) \\ &= \int_{\mathbb{R}^3} \rho u \cdot \nabla u \cdot \dot{u} + \frac{d}{dt} \int_{\mathbb{R}^3} P \operatorname{div} u - \int_{\mathbb{R}^3} P_t \operatorname{div} u \\ &= \frac{d}{dt} \int_{\mathbb{R}^3} P \operatorname{div} u - \frac{1}{2(2\mu + \lambda)} \frac{d}{dt} \int_{\mathbb{R}^3} P^2 - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} P_t G + \int_{\mathbb{R}^3} \rho u \cdot \nabla u \cdot \dot{u} \\ &= \frac{d}{dt} \int_{\mathbb{R}^3} P \operatorname{div} u - \frac{1}{2(2\mu + \lambda)} \frac{d}{dt} \int_{\mathbb{R}^3} P^2 + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} (\operatorname{div}(Pu) + (\gamma - 1) P \operatorname{div} u) G \\ &\quad + \int_{\mathbb{R}^3} \rho u \cdot \nabla u \cdot \dot{u} \\ &= \frac{d}{dt} \int_{\mathbb{R}^3} P \operatorname{div} u - \frac{1}{2(2\mu + \lambda)} \frac{d}{dt} \int_{\mathbb{R}^3} P^2 - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} Pu \cdot \nabla G + \frac{\gamma - 1}{2\mu + \lambda} \int_{\mathbb{R}^3} P \operatorname{div} u G \\ &\quad + \int_{\mathbb{R}^3} \rho u \cdot \nabla u \cdot \dot{u} = \sum_{i=1}^5 I_i, \end{aligned} \quad (3.21)$$

where $G = (2\mu + \lambda) \operatorname{div} u - P$.

For I_3 , using Hölder inequality, we have

$$I_3 \lesssim \int_{\mathbb{R}^3} P |u| |\nabla G| \lesssim \|\rho^{\frac{1}{r}} u\|_{L^r} \left\| \rho^{\gamma - \frac{1}{r}} \right\|_{L^{\frac{rp_1}{r p_1 - p_1 - r}}} \|\nabla G\|_{L^{p_1}}, \quad (3.22)$$

for some $p_1 \in (1, 2)$.

Taking div on both side of (2.2.1)₂, we have

$$\Delta G = \operatorname{div}(\rho \dot{u}). \quad (3.23)$$

From the standard elliptic estimates together with (3.1), we have

$$\|\nabla G\|_{L^{p_1}} \lesssim \|\rho \dot{u}\|_{L^{p_1}} \lesssim \|\sqrt{\rho} \dot{u}\|_{L^2} \left\| \sqrt{\rho} \right\|_{L^{\frac{2p_1}{2-p_1}}} \lesssim \|\sqrt{\rho} \dot{u}\|_{L^2}. \quad (3.24)$$

By (3.22), (3.24), Lemma 3.1 and (3.1), we have

$$I_3 \leq C \|\sqrt{\rho}\dot{u}\|_{L^2}. \quad (3.25)$$

For I_4 , we have

$$\begin{aligned} I_4 &\lesssim \int_{\mathbb{R}^3} P |\operatorname{div} u| |G| \\ &\lesssim \|P\|_{L^{\frac{6p_1}{5p_1-6}}} \|\operatorname{div} u\|_{L^2} \|G\|_{L^{\frac{3p_1}{3-p_1}}} \\ &\lesssim \|\operatorname{div} u\|_{L^2} \|\nabla G\|_{L^{p_1}}, \end{aligned} \quad (3.26)$$

where we have used Hölder inequality, Sobolev inequality and (3.1).

Substituting (3.24) into (3.26), we have

$$I_4 \leq C \|\operatorname{div} u\|_{L^2} \|\sqrt{\rho}\dot{u}\|_{L^2}. \quad (3.27)$$

For I_5 , we have

$$I_5 \leq \|\sqrt{\rho}\dot{u}\|_{L^2} \|\sqrt{\rho}u \cdot \nabla u\|_{L^2}. \quad (3.28)$$

Assume $p_2 \in (\frac{2r}{r-2}, 6)$, and let $\frac{3p_2}{3+p_2} < p_1$, we have for any $\varepsilon \in (0, 1)$

$$\begin{aligned} \|\sqrt{\rho}u \cdot \nabla u\|_{L^2} &\leq \|\rho^{\frac{1}{r}}u\|_{L^r} \left\| \rho^{\frac{1}{2}-\frac{1}{r}} \right\|_{L^{\frac{2rp_2}{rp_2-2p_2-2r}}} \|\nabla u\|_{L^{p_2}} \\ &\lesssim \|\operatorname{div} u\|_{L^{p_2}} + \|\operatorname{curl} u\|_{L^{p_2}} \\ &\lesssim \|G\|_{L^{p_2}} + \|\operatorname{curl} u\|_{L^{p_2}} + 1 \\ &\leq \varepsilon \|\nabla G\|_{L^{p_1}} + \varepsilon \|\nabla \operatorname{curl} u\|_{L^{p_1}} + C_\varepsilon \|\nabla u\|_{L^2} + C, \end{aligned} \quad (3.29)$$

where we have used Hölder inequality, (3.1), Lemma 3.1, and the standard interpolation inequality.

Taking curl on both side of (2.2.1)₂, we have

$$\mu \Delta(\operatorname{curl} u) = \operatorname{curl}(\rho \dot{u}).$$

Similar to (3.24), we have

$$\|\nabla \operatorname{curl} u\|_{L^{p_1}} \lesssim \|\sqrt{\rho}\dot{u}\|_{L^2}. \quad (3.30)$$

Substituting (3.24) and (3.30) into (3.29), we have

$$\|\sqrt{\rho}u \cdot \nabla u\|_{L^2} \leq \varepsilon C \|\sqrt{\rho}\dot{u}\|_{L^2} + C_\varepsilon \|\nabla u\|_{L^2} + C. \quad (3.31)$$

Substituting (3.31) into (3.28), we have

$$I_5 \leq \varepsilon C \|\sqrt{\rho}\dot{u}\|_{L^2}^2 + C_\varepsilon \|\nabla u\|_{L^2}^2 + C. \quad (3.32)$$

Putting (3.25), (3.27) and (3.32) into (3.21), using Cauchy inequality, and taking ε sufficiently small, we have

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + (\mu + \lambda) |\operatorname{div} u|^2) \\ &\leq \frac{d}{dt} \int_{\mathbb{R}^3} P \operatorname{div} u - \frac{1}{2(2\mu + \lambda)} \frac{d}{dt} \int_{\mathbb{R}^3} P^2 + C \|\nabla u\|_{L^2}^2 + C. \end{aligned} \quad (3.33)$$

Integrating (3.33) over $[0, t]$, and using Cauchy inequality, we have

$$\begin{aligned} & \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \frac{1}{2} \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + (\mu + \lambda) |\operatorname{div} u|^2) \\ & \leq \int_{\mathbb{R}^3} P \operatorname{div} u + C \int_0^t \|\nabla u\|_{L^2}^2 + C \leq \frac{\mu + \lambda}{2} \int_{\mathbb{R}^3} |\operatorname{div} u|^2 + C \int_{\mathbb{R}^3} P^2 + C \int_0^t \|\nabla u\|_{L^2}^2 + C. \end{aligned}$$

This together with Gronwall inequality gives

$$\int_0^t \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \int_{\mathbb{R}^3} |\nabla u|^2 \leq C,$$

for any $t \in [0, T^*)$. \square

Lemma 3.4 *Under the conditions of Theorem 2.2.2 and (3.1), it holds that for any $T \in [0, T^*)$*

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \int_0^T \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 \leq C.$$

Proof. By the definition of \dot{u} , we can write (2.2.1)₂ as follows:

$$\rho \dot{u} + \nabla(P(\rho)) = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u. \quad (3.34)$$

Differentiating (3.34) with respect to t and using (2.2.1)₁, we have

$$\begin{aligned} \rho \dot{u}_t + \rho u \cdot \nabla \dot{u} + \nabla P_t &= \mu \Delta \dot{u} + (\mu + \lambda) \nabla \operatorname{div} \dot{u} - \mu \Delta(u \cdot \nabla u) - (\mu + \lambda) \nabla \operatorname{div}(u \cdot \nabla u) \\ &+ \operatorname{div}(\mu \Delta u \otimes u + (\mu + \lambda) \nabla \operatorname{div} u \otimes u - \nabla P \otimes u). \end{aligned} \quad (3.35)$$

Multiplying (3.35) by \dot{u} , integrating by parts over \mathbb{R}^3 , for $t \in (0, T^*)$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \int_{\mathbb{R}^3} (\mu |\nabla \dot{u}|^2 + (\mu + \lambda) |\operatorname{div} \dot{u}|^2) \\ &= \int_{\mathbb{R}^3} (P_t \operatorname{div} \dot{u} + u \otimes \nabla P : \nabla \dot{u}) + \mu \int_{\mathbb{R}^3} (\operatorname{div}(\Delta u \otimes u) - \Delta(u \cdot \nabla u)) \cdot \dot{u} \\ &+ (\mu + \lambda) \int_{\mathbb{R}^3} (\operatorname{div}(\nabla \operatorname{div} u \otimes u) - \nabla \operatorname{div}(u \cdot \nabla u)) \cdot \dot{u} = \sum_{i=1}^3 II_i. \end{aligned} \quad (3.36)$$

For II_1 , using (2.2.1)₁, we have

$$\begin{aligned} II_1 &= \int_{\mathbb{R}^3} \left(-\operatorname{div}(Pu) \operatorname{div} \dot{u} - (\gamma - 1) P \operatorname{div} u \operatorname{div} \dot{u} + u \otimes \nabla P : \nabla \dot{u} \right) \\ &= \int_{\mathbb{R}^3} \left(Pu \cdot \nabla \operatorname{div} \dot{u} - (\gamma - 1) P \operatorname{div} u \operatorname{div} \dot{u} - P(\nabla u)^t : \nabla \dot{u} - Pu \cdot \nabla \operatorname{div} \dot{u} \right) \\ &= - \int_{\mathbb{R}^3} \left((\gamma - 1) P \operatorname{div} u \operatorname{div} \dot{u} + P(\nabla u)^t : \nabla \dot{u} \right) \lesssim \|P\|_{L^4} \|\nabla u\|_{L^4} \|\nabla \dot{u}\|_{L^2}. \end{aligned} \quad (3.37)$$

For II_2 and II_3 , we use the similar arguments as [18, 20, 36, 37]. More precisely, we have

$$\operatorname{div}(\Delta u \otimes u) - \Delta(u \cdot \nabla u) = \nabla_k(\operatorname{div} u \nabla_k u) - \nabla_k(\nabla_k u^j \nabla_j u) - \nabla_j(\nabla_k u^j \nabla_k u).$$

Using integration by parts, we have

$$II_2 = \mu \int \left(\nabla_k (\operatorname{div} u \nabla_k u) - \nabla_k (\nabla_k u^j \nabla_j u) - \nabla_j (\nabla_k u^j \nabla_k u) \right) \cdot \dot{u} \lesssim \|\nabla \dot{u}\|_{L^2} \|\nabla u\|_{L^4}^2. \quad (3.38)$$

Similarly, since

$$\operatorname{div} (\nabla \operatorname{div} u \otimes u) - \nabla \operatorname{div} (u \cdot \nabla u) = \nabla (\nabla_j u^j \nabla_i u^i) - \nabla (\nabla_j u^i \nabla_i u^j) - \nabla_i (\nabla u^i \nabla_j u^j),$$

we have

$$II_3 = (\mu + \lambda) \int (\nabla (\nabla_j u^j \nabla_i u^i) - \nabla (\nabla_j u^i \nabla_i u^j) - \nabla_i (\nabla u^i \nabla_j u^j)) \cdot \dot{u} \lesssim \|\nabla \dot{u}\|_{L^2} \|\nabla u\|_{L^4}^2. \quad (3.39)$$

Substituting (3.37), (3.38) and (3.39) into (3.36), and using Cauchy inequality and (3.1), we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \int_{\mathbb{R}^3} (\mu |\nabla \dot{u}|^2 + (\mu + \lambda) |\operatorname{div} \dot{u}|^2) \leq \frac{\mu}{2} \|\nabla \dot{u}\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4 + C.$$

This gives

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \mu \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 \\ & \leq C \|\nabla u\|_{L^4}^4 + C \lesssim \|\operatorname{div} u\|_{L^4}^4 + \|\operatorname{curl} u\|_{L^4}^4 + 1 \\ & \lesssim \|G\|_{L^4}^4 + \|\operatorname{curl} u\|_{L^4}^4 + 1 \\ & \lesssim \|G\|_{L^2}^{\frac{2(7p_1-12)}{5p_1-6}} \|\nabla G\|_{L^{p_1}}^{\frac{6p_1}{5p_1-6}} + \|\operatorname{curl} u\|_{L^2}^{\frac{2(7p_1-12)}{5p_1-6}} \|\nabla \operatorname{curl} u\|_{L^{p_1}}^{\frac{6p_1}{5p_1-6}} + 1, \end{aligned} \quad (3.40)$$

where we have used Gagliardo-Nirenberg inequality and (3.1).

By (3.40), Lemma 3.3, (3.1), (3.24), (3.30) and Young inequality, we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \mu \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 \lesssim \|\sqrt{\rho} \dot{u}\|_{L^2}^{\frac{6p_1}{5p_1-6}} + 1 \lesssim \|\sqrt{\rho} \dot{u}\|_{L^2}^4 + 1, \quad (3.41)$$

where we have used the fact $\frac{6p_1}{5p_1-6} < 4$, since $p_1 > \frac{3p_2}{3+p_2} \geq \frac{12}{7}$.

Since $\|\sqrt{\rho} \dot{u}\|_{L^2}^2$ is bounded in $L^1(0, T)$ (see Lemma 3.3), we apply (3.41) and Gronwall inequality to complete the proof of Lemma 3.4. \square

Corollary 3.5 *Under the conditions of Theorem 2.2.2 and (3.1), it holds that for any $T \in [0, T^*)$*

$$\|\nabla G\|_{L^2(0, T; L^{\frac{6p_1}{12-5p_1}})} \leq C.$$

Proof. By (3.23) and the standard elliptic estimates, together with (3.1), Hölder inequality, Sobolev inequality and Lemma 3.4, we have for any $T \in [0, T^*)$

$$\int_0^T \|\nabla G\|_{L^{\frac{6p_1}{12-5p_1}}}^2 \leq C \int_0^T \|\rho \dot{u}\|_{L^{\frac{6p_1}{12-5p_1}}}^2 \leq C \int_0^T \|\rho\|_{L^{\frac{p_1}{2-5p_1}}}^2 \|\dot{u}\|_{L^6}^2 \leq C \int_0^T \|\nabla \dot{u}\|_{L^2}^2 \leq C.$$

\square

Lemma 3.6 *Under the conditions of Theorem 2.2.2 and (3.1), it holds that for any $T \in [0, T^*)$*

$$\|\rho\|_{L^\infty(0, T; L^\infty)} \leq C.$$

Proof. For any $1 < p < +\infty$, multiplying (2.2.1)₁ by $p\rho^{p-1}$ and integrating by parts over \mathbb{R}^3 , we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} \rho^p &= - \int_{\mathbb{R}^3} \left(u \cdot \nabla(\rho^p) + p\rho^p \operatorname{div} u \right) \\ &= (1-p) \int_{\mathbb{R}^3} \rho^p \operatorname{div} u = \frac{1-p}{2\mu+\lambda} \int_{\mathbb{R}^3} \rho^p G + \frac{1-p}{2\mu+\lambda} \int_{\mathbb{R}^3} \rho^p P \\ &\leq \frac{p-1}{2\mu+\lambda} \|G\|_{L^\infty} \int_{\mathbb{R}^3} \rho^p. \end{aligned} \quad (3.42)$$

Since $\frac{6p_1}{12-5p_1} > 3$, using the standard interpolation inequality, we have

$$\|G\|_{L^\infty} \leq C \left\| \nabla G \right\|_{L^{\frac{6p_1}{12-5p_1}}} + C \|G\|_{L^2} \leq C \left\| \nabla G \right\|_{L^{\frac{6p_1}{12-5p_1}}} + C. \quad (3.43)$$

Substituting (3.43) into (3.42), we have

$$\begin{aligned} \frac{d}{dt} \|\rho\|_{L^p} &\leq \frac{C(p-1)}{p} \left(\left\| \nabla G \right\|_{L^{\frac{6p_1}{12-5p_1}}} + 1 \right) \|\rho\|_{L^p} \\ &\leq C \left(\left\| \nabla G \right\|_{L^{\frac{6p_1}{12-5p_1}}} + 1 \right) \|\rho\|_{L^p}, \end{aligned}$$

where the constant C is independent of p . This and Corollary 3.5, together with Gronwall inequality, give

$$\sup_{0 \leq t \leq T} \|\rho(t)\|_{L^p} \leq \|\rho_0\|_{L^p} \exp \left(C \int_0^T \left(\left\| \nabla G \right\|_{L^{\frac{6p_1}{12-5p_1}}} + 1 \right) dt \right) \leq C,$$

for any $T \in [0, T^*)$. Let p go to ∞ , we complete the proof of Lemma 3.6. \square

4 Proof of Theorem 2.3.2

Let $0 < T^* < \infty$ be the maximum time of existence of strong solution (ρ, u) to (2.3.1)-(2.3.3). Namely, (ρ, u) is a strong solution to (2.3.1)-(2.3.3) in $\mathbb{R}^3 \times [0, T]$ for any $0 < T < T^*$, but not a strong solution in $\mathbb{R}^3 \times [0, T^*]$. We shall prove Theorem 2.3.2 by using a contradiction argument. Suppose that (2.3.5) were false, i.e.

$$M := \|\rho\|_{L^\infty(0, T^*; L^\infty)} + \|\theta\|_{L^\infty(0, T^*; L^\infty)} < \infty. \quad (4.1)$$

The goal is to show that under the assumption (4.1), there is a bound $C > 0$ depending only on $M, \rho_0, u_0, \theta_0, \mu, \lambda, \kappa$, and T^* such that

$$\sup_{0 \leq t < T^*} \left[\max_{l=2, q} (\|\rho\|_{W^{1, l}} + \|\rho_t\|_{L^l}) + \|(\sqrt{\rho}u_t, \sqrt{\rho}\theta_t)\|_{L^2} + \|(\nabla u, \nabla \theta)\|_{H^1} \right] \leq C, \quad (4.2)$$

and

$$\int_0^{T^*} (\|(u_t, \theta_t)\|_{D^1}^2 + \|(u, \theta)\|_{D^{2, q}}^2) dt \leq C. \quad (4.3)$$

With (4.2) and (4.3), it is easy to show without much difficulties that T^* is not the maximum time, which is the desired contradiction.

Throughout the rest of the section, we denote by C a generic constant depending only on $\rho_0, u_0, \theta_0, T^*, M, \lambda, \mu, \kappa$. We denote by

$$A \lesssim B$$

if there exists a generic constant C such that $A \leq CB$.

Lemma 4.1 *Under the conditions of Theorem 2.3.2 and (4.1), it holds that*

$$\begin{cases} \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \rho(|u|^2 + \theta^2) dx + \int_0^T \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla \theta|^2) dx \leq C, \\ \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \rho \leq C, \text{ for any } T \in [0, T^*]. \end{cases} \quad (4.4)$$

Proof. The proof of (4.4)₁ can be referred to [37] (Lemma 2). (4.4)₂ can be obtained by integrating (2.3.1)₁ over $\mathbb{R}^3 \times [0, t]$. \square

Lemma 4.2 *Under the conditions of Theorem 2.3.2 and (4.1), if $3\mu > \lambda$, it holds that*

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \rho|u|^4 + \int_0^T \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 dx \leq C, \quad (4.5)$$

for any $T \in [0, T^*)$.

Proof. The proof of the lemma is quite similar to that of Lemma 3.1 except that $r = 4$ and $P = \rho\theta$ here. From (3.4), we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \rho|u|^r + \int_{\mathbb{R}^3 \cap \{|u|>0\}} r|u|^{r-2} (\mu|\nabla u|^2 + (\lambda + \mu)|\operatorname{div} u|^2 + \mu(r-2)|\nabla|u||^2) \\ &= r \int_{\mathbb{R}^3 \cap \{|u|>0\}} \operatorname{div}(|u|^{r-2} u) P - r(r-2)(\mu + \lambda) \int_{\mathbb{R}^3 \cap \{|u|>0\}} \operatorname{div} u |u|^{r-3} u \cdot \nabla|u|. \end{aligned} \quad (4.6)$$

For any given $\varepsilon_1 \in (0, 1)$, we define a function as in the proof of Lemma 3.1 as follows:

$$\phi(\varepsilon_1, r) = \begin{cases} \frac{\mu\varepsilon_1(r-1)}{3\left(-\frac{4\mu}{3} - \lambda + \frac{r^2(\mu+\lambda)}{4(r-1)}\right)}, & \text{if } \frac{r^2(\mu+\lambda)}{4(r-1)} - \frac{4\mu}{3} - \lambda > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Case 1:

$$\int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^r \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 > \phi(\varepsilon_1, r) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla|u||^2. \quad (4.7)$$

Using the similar arguments like in the proof of Lemma 3.1, for any $\varepsilon_0 \in (0, 1)$, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \rho|u|^r + \int_{\mathbb{R}^3 \cap \{|u|>0\}} \mu r |u|^{r-2} |\nabla|u||^2 + \int_{\mathbb{R}^3 \cap \{|u|>0\}} \mu r |u|^r \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 \\ &+ \mu(r-2)r \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla|u||^2 \\ &\leq C \int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho|u|^{r-2} |\nabla|u|| + \mu r \varepsilon_0 \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^r \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 \\ &+ \frac{C}{4\mu r \varepsilon_0} \left(\int_{\mathbb{R}^3} \rho|u|^r \right)^{\frac{r-2}{r}} \left(\int_{\mathbb{R}^3} \rho^{\frac{r}{2}+1} \right)^{\frac{2}{r}} + \frac{r(r-2)^2(\mu + \lambda)}{4} \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla|u||^2. \end{aligned}$$

Combining (4.1) and (4.7), we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \rho|u|^r + r f(\varepsilon_0, \varepsilon_1, \varepsilon_2, r) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla|u||^2 + \mu r (1 - \varepsilon_0) \varepsilon_2 \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^r \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 \\ &\leq C \int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho|u|^{r-2} |\nabla|u|| + \frac{C}{4\mu r \varepsilon_0} \left(\int_{\mathbb{R}^3} \rho|u|^r \right)^{\frac{r-2}{r}}, \end{aligned} \quad (4.8)$$

where

$$f(\varepsilon_0, \varepsilon_1, \varepsilon_2, r) = \mu(1 - \varepsilon_0)(1 - \varepsilon_2)\phi(\varepsilon_1, r) + \mu(r - 1) - \frac{(r - 2)^2(\mu + \lambda)}{4}, \quad (4.9)$$

for $\varepsilon_2 \in (0, 1)$ decided later.

(Sub-Case 1₁): If $4 \in \{r | \frac{r^2(\mu + \lambda)}{4(r-1)} - \frac{4\mu}{3} - \lambda > 0\}$, i.e., $\lambda > 0$, we have

$$\phi(\varepsilon_1, 4) = \frac{3\mu\varepsilon_1}{\lambda}. \quad (4.10)$$

Substituting (4.10) into (4.9), we have

$$f(\varepsilon_0, \varepsilon_1, \varepsilon_2, r) = \frac{\mu^2\varepsilon_1(1 - \varepsilon_0)(1 - \varepsilon_2)(r - 1)}{3\left(-\frac{4\mu}{3} - \lambda + \frac{r^2(\mu + \lambda)}{4(r-1)}\right)} + \mu(r - 1) - \frac{(r - 2)^2(\mu + \lambda)}{4}.$$

For $(\varepsilon_0, \varepsilon_1, \varepsilon_2, r) = (0, 1, 0, 4)$, we have

$$f(0, 1, 0, 4) = \frac{3\mu^2}{\lambda} + 2\mu - \lambda > 0,$$

where we have used $0 < \lambda < 3\mu$.

Since $f(\varepsilon_0, \varepsilon_1, \varepsilon_2, 4)$ is continuous w.r.t. $(\varepsilon_0, \varepsilon_1, \varepsilon_2)$ over $[0, 1] \times [0, 1] \times [0, 1]$, there exist $\varepsilon_0, \varepsilon_1, \varepsilon_2 \in (0, 1)$ such that

$$f(\varepsilon_0, \varepsilon_1, \varepsilon_2, 4) > 0.$$

By (4.8), Cauchy inequality and Hölder inequality, for $r = 4$, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^4 + 4f(\varepsilon_0, \varepsilon_1, \varepsilon_2, 4) \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |u|^2 |\nabla |u||^2 + 4\mu(1 - \varepsilon_0)\varepsilon_2 \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |u|^4 \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 \\ & \leq 2f(\varepsilon_0, \varepsilon_1, \varepsilon_2, 4) \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |u|^2 |\nabla |u||^2 + \frac{C}{f(\varepsilon_0, \varepsilon_1, \varepsilon_2, 4)} \left(\int_{\mathbb{R}^3} \rho |u|^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \rho^3 \right)^{\frac{1}{2}} \\ & \quad + \frac{C}{\varepsilon_0} \left(\int_{\mathbb{R}^3} \rho |u|^4 \right)^{\frac{1}{2}}. \end{aligned}$$

This together with (4.1) gives

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^4 + 2f(\varepsilon_0, \varepsilon_1, \varepsilon_2, 4) \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |u|^2 |\nabla |u||^2 + 4\mu(1 - \varepsilon_0)\varepsilon_2 \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |u|^4 \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 \\ & \leq C \left[\frac{1}{f(\varepsilon_0, \varepsilon_1, \varepsilon_2, 4)} + \frac{1}{\varepsilon_0} \right] \left(\int_{\mathbb{R}^3} \rho |u|^4 \right)^{\frac{1}{2}}. \end{aligned} \quad (4.11)$$

(Sub-Case 1₂): if $4 \notin \{r | \frac{r^2(\mu + \lambda)}{4(r-1)} - \frac{4\mu}{3} - \lambda > 0\}$, i.e., $\lambda \leq 0$, we have $\phi(\varepsilon_1, 4) = 0$.

In this case, it is easy to get

$$4f(\varepsilon_0, \varepsilon_1, \varepsilon_2, 4) = 4(2\mu - \lambda) \geq 8\mu. \quad (4.12)$$

By (4.8) (for $r = 4$), (4.12), Cauchy inequality and Hölder inequality, we have

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^4 + 8\mu \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^2 |\nabla |u||^2 + 4\mu(1 - \varepsilon_0)\varepsilon_2 \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^4 \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 \\
& \leq C \int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho |u|^2 |\nabla |u|| + \frac{C}{\varepsilon_0} \left(\int_{\mathbb{R}^3} \rho |u|^4 \right)^{\frac{1}{2}} \\
& \leq 4\mu \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^2 |\nabla |u||^2 + C \left(\int_{\mathbb{R}^3} \rho |u|^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \rho^3 \right)^{\frac{1}{2}} + \frac{C}{\varepsilon_0} \left(\int_{\mathbb{R}^3} \rho |u|^4 \right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^4 + 4\mu \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^2 |\nabla |u||^2 + 4\mu(1 - \varepsilon_0)\varepsilon_2 \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^4 \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 \\
& \leq C \left(\int_{\mathbb{R}^3} \rho |u|^4 \right)^{\frac{1}{2}},
\end{aligned} \tag{4.13}$$

where we have used (4.1).

Case 2: if

$$\int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^r \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 \leq \phi(\varepsilon_1, r) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2. \tag{4.14}$$

Using the similar arguments like in the proof of Lemma 3.1, we have

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r + \left[3r \left(\frac{4\mu}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r-1)} \right) \phi(\varepsilon_1, r) + \mu r(r-1) \right] \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 \\
& \leq C \int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho^{1-\frac{r-2}{2r}} \rho^{\frac{r-2}{2r}} |u|^{r-2} |\nabla |u|| \\
& \leq \varepsilon \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 + \frac{C}{\varepsilon} \left(\int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho |u|^r \right)^{\frac{r-2}{r}} \left(\int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho^{\frac{r}{2}+1} \right)^{\frac{2}{r}} \\
& \leq \varepsilon(1 + \phi(\varepsilon_1, r)) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 + \frac{C}{\varepsilon} \left(\int_{\mathbb{R}^3 \cap \{|u|>0\}} \rho |u|^r \right)^{\frac{r-2}{r}},
\end{aligned}$$

where we have used Cauchy inequality, Hölder inequality and (4.1).

Taking $\varepsilon = (2 + 2\phi(\varepsilon_1, r))^{-1} \left[3r \left(\frac{4\mu}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r-1)} \right) \phi(\varepsilon_1, r) + \mu r(r-1) \right]$, and using (4.14) and (3.6), we have

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^r + \frac{3r \left(\frac{4\mu}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r-1)} \right) \phi(\varepsilon_1, r) + \mu r(r-1)}{2(1 + \phi(\varepsilon_1, r))} \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} |\nabla |u||^2 \\
& \leq \frac{C(1 + \phi(\varepsilon_1, r))}{3r \left(\frac{4\mu}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r-1)} \right) \phi(\varepsilon_1, r) + \mu r(r-1)} \left(\int_{\mathbb{R}^3} \rho |u|^r \right)^{\frac{r-2}{r}},
\end{aligned} \tag{4.15}$$

for $r = 4$.

By (3.6), (4.11), (4.13), (4.15) and Cauchy inequality, for **Case 1** and **Case 2**, we conclude that if $3\mu > \lambda$, there exists a constant $c_1 > 0$ such that

$$\frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^4 + c_1 \int_{\mathbb{R}^3 \cap \{|u| > 0\}} |u|^2 |\nabla u|^2 \leq C \int_{\mathbb{R}^3} \rho |u|^4 + C, \quad (4.16)$$

for $t \in [0, T^*)$. By (4.16) and Gronwall inequality, we get (4.5). \square

Lemma 4.3 *Under the conditions of Theorem 2.3.2 and (4.1), it holds that for any $T \in [0, T^*)$*

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_0^T \int_{\mathbb{R}^3} \rho |u_t|^2 dx dt \leq C. \quad (4.17)$$

Proof. Multiplying (2.3.1)₂ by u_t , and integrating by parts over \mathbb{R}^3 , we have

$$\begin{aligned} & \int_{\mathbb{R}^3} \rho |u_t|^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + (\mu + \lambda) |\operatorname{div} u|^2) \\ &= - \int_{\mathbb{R}^3} \rho u \cdot \nabla u \cdot u_t + \frac{d}{dt} \int_{\mathbb{R}^3} P \operatorname{div} u - \int_{\mathbb{R}^3} P_t \operatorname{div} u \\ &= \frac{d}{dt} \int_{\mathbb{R}^3} P \operatorname{div} u - \frac{1}{2(2\mu + \lambda)} \frac{d}{dt} \int_{\mathbb{R}^3} P^2 - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} P_t G - \int_{\mathbb{R}^3} \rho u \cdot \nabla u \cdot u_t \\ &= \sum_{i=1}^4 III_i, \end{aligned} \quad (4.18)$$

where $G = (2\mu + \lambda) \operatorname{div} u - P$.

For III_3 , recalling $\rho E = P + \frac{\rho |u|^2}{2}$, we have

$$\begin{aligned} III_3 &= - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} (\rho E)_t G + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \left(\frac{\rho |u|^2}{2} \right)_t G \\ &= \sum_{i=1}^2 III_{3,i}. \end{aligned} \quad (4.19)$$

For $III_{3,1}$, using (1.1)₃, integration by parts, (4.1) and (4.4), we have

$$\begin{aligned} III_{3,1} &= - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta u \cdot \nabla G - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \frac{|u|^2}{2} u \cdot \nabla G - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} P u \cdot \nabla G \\ &\quad + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \tau u \nabla G + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \nabla \theta \cdot \nabla G \\ &\leq - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \frac{|u|^2}{2} u \cdot \nabla G + C \|\nabla G\|_{L^2} (\|\rho \theta u\|_{L^2} + \|P u\|_{L^2} + \|u |\nabla u|\|_{L^2} + \|\nabla \theta\|_{L^2}) \\ &\leq - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \frac{|u|^2}{2} u \cdot \nabla G + C \|\nabla G\|_{L^2} (\|u |\nabla u|\|_{L^2} + \|\nabla \theta\|_{L^2} + 1). \end{aligned} \quad (4.20)$$

Taking div and curl on both side of (2.3.1)₂, we get

$$\Delta G = \operatorname{div}(\rho u_t + \rho u \cdot \nabla u), \quad (4.21)$$

and

$$\mu\Delta(\operatorname{curl}u) = \operatorname{curl}(\rho u_t + \rho u \cdot \nabla u). \quad (4.22)$$

From the standard elliptic estimates together with (4.1), we get

$$\|\nabla G\|_{L^2} \lesssim \|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} \lesssim \|\sqrt{\rho}u_t\|_{L^2} + \|\sqrt{\rho}|u|\nabla u\|_{L^2}, \quad (4.23)$$

and

$$\|\nabla \operatorname{curl}u\|_{L^2} \lesssim \|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} \lesssim \|\sqrt{\rho}u_t\|_{L^2} + \|\sqrt{\rho}|u|\nabla u\|_{L^2}. \quad (4.24)$$

To handle the second term of the right hand side of (4.23) and (4.24), we use the fact

$$-\Delta f = \nabla \times (\operatorname{curl}f) - \nabla \operatorname{div}f, \text{ in } \mathbb{R}^3, \quad (4.25)$$

for some $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Using (4.25) and the elliptic estimates, we have

$$\|\nabla f\|_{L^p} \lesssim \|\operatorname{curl}f\|_{L^p} + \|\operatorname{div}f\|_{L^p}, \quad (4.26)$$

for any $p \in (1, \infty)$. Let's go back to handle $\|\sqrt{\rho}|u|\nabla u\|_{L^2}$. Using Hölder inequality, (4.1), (4.5), (4.26) for $p = 4$, Gagliardo-Nirenberg inequality, (4.23), (4.24) and Cauchy inequality, we have

$$\begin{aligned} \|\sqrt{\rho}|u|\nabla u\|_{L^2} &\lesssim \|\rho^{\frac{1}{4}}u\|_{L^4} \|\nabla u\|_{L^4} \lesssim \|\operatorname{curl}u\|_{L^4} + \|\operatorname{div}u\|_{L^4} \\ &\lesssim \|\operatorname{curl}u\|_{L^4} + \|G\|_{L^4} + 1 \\ &\leq C \|\operatorname{curl}u\|_{L^2}^{\frac{1}{4}} \|\nabla \operatorname{curl}u\|_{L^2}^{\frac{3}{4}} + C \|G\|_{L^2}^{\frac{1}{4}} \|\nabla G\|_{L^2}^{\frac{3}{4}} + C \\ &\leq C \|\operatorname{curl}u\|_{L^2}^{\frac{1}{4}} \|\sqrt{\rho}u_t\|_{L^2}^{\frac{3}{4}} + C \|G\|_{L^2}^{\frac{1}{4}} \|\sqrt{\rho}u_t\|_{L^2}^{\frac{3}{4}} + \frac{1}{2} \|\sqrt{\rho}|u|\nabla u\|_{L^2} + C \|\nabla u\|_{L^2} + C. \end{aligned}$$

This, together with Young inequality, gives

$$\|\sqrt{\rho}|u|\nabla u\|_{L^2} \leq \epsilon \|\sqrt{\rho}u_t\|_{L^2} + C_\epsilon \|\nabla u\|_{L^2} + C, \quad (4.27)$$

for any $\epsilon > 0$. Substituting (4.27) into (4.23), we have

$$\|\nabla G\|_{L^2} \lesssim \|\sqrt{\rho}u_t\|_{L^2} + \|\nabla u\|_{L^2} + 1. \quad (4.28)$$

Substituting (4.28) into (4.20), and using Cauchy inequality, we have

$$\begin{aligned} III_{3,1} &\leq -\frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \frac{|u|^2}{2} u \cdot \nabla G + \frac{1}{6} \|\sqrt{\rho}u_t\|_{L^2}^2 + C \|u|\nabla u\|_{L^2}^2 \\ &\quad + C \|\nabla \theta\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 + C. \end{aligned} \quad (4.29)$$

For $III_{3,2}$, we have

$$\begin{aligned} III_{3,2} &= \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \frac{\rho_t |u|^2}{2} G + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho u \cdot u_t G \\ &\leq -\frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \frac{\operatorname{div}(\rho u) |u|^2}{2} G + \frac{1}{24} \int_{\mathbb{R}^3} \rho |u_t|^2 + C \int_{\mathbb{R}^3} \rho |u|^2 |G|^2 \\ &\leq \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho u \cdot \nabla u \cdot u G + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \frac{\rho u |u|^2}{2} \cdot \nabla G + \frac{1}{24} \int_{\mathbb{R}^3} \rho |u_t|^2 \\ &\quad + C \int_{\mathbb{R}^3} \rho |u|^2 |\nabla u|^2 + C \\ &\leq C \int_{\mathbb{R}^3} \rho |u|^2 |\nabla u|^2 + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \frac{\rho u |u|^2}{2} \cdot \nabla G + \frac{1}{12} \int_{\mathbb{R}^3} \rho |u_t|^2 + C. \end{aligned} \quad (4.30)$$

Using (4.27) again (for $\epsilon > 0$ sufficiently small), together with (4.30), Lemma 4.2, (4.28) and Cauchy inequality, we get

$$\begin{aligned} III_{3,2} &\leq \frac{1}{12} \int_{\mathbb{R}^3} \rho |u_t|^2 + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \frac{\rho u |u|^2}{2} \cdot \nabla G + \frac{1}{12} \int_{\mathbb{R}^3} \rho |u_t|^2 + C \int_{\mathbb{R}^3} |\nabla u|^2 + C \\ &= \frac{1}{6} \int_{\mathbb{R}^3} \rho |u_t|^2 + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \frac{\rho u |u|^2}{2} \cdot \nabla G + C \int_{\mathbb{R}^3} |\nabla u|^2 + C. \end{aligned} \quad (4.31)$$

Substituting (4.29) and (4.31) into (4.19), we have

$$III_3 \leq \frac{1}{3} \int_{\mathbb{R}^3} \rho |u_t|^2 + C \|u |\nabla u|\|_{L^2}^2 + C \|\nabla \theta\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 + C. \quad (4.32)$$

For III_4 , using Cauchy inequality and (4.27) (for $\epsilon > 0$ sufficiently small), we have

$$\begin{aligned} III_4 &\leq \frac{1}{12} \int_{\mathbb{R}^3} \rho |u_t|^2 + C \int_{\mathbb{R}^3} \rho |u|^2 |\nabla u|^2 \\ &\leq \frac{1}{6} \int_{\mathbb{R}^3} \rho |u_t|^2 + C \int_{\mathbb{R}^3} |\nabla u|^2 + C. \end{aligned} \quad (4.33)$$

Putting (4.32) and (4.33) into (4.18), and integrating it over $[0, t]$, for $t < T^*$, we have

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^3} \rho |u_t|^2 + \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + (\mu + \lambda) |\operatorname{div} u|^2) \\ &\leq 2 \int_{\mathbb{R}^3} P \operatorname{div} u + C \int_0^t \|u |\nabla u|\|_{L^2}^2 + C \int_0^t \|\nabla \theta\|_{L^2}^2 + C \int_0^t \|\nabla u\|_{L^2}^2 + C \\ &\leq (\mu + \lambda) \int_{\mathbb{R}^3} |\operatorname{div} u|^2 + C, \end{aligned}$$

where we have used Cauchy inequality, (4.1), Lemmas 4.1 and 4.2.

Therefore,

$$\int_0^t \int_{\mathbb{R}^3} \rho |u_t|^2 + \int_{\mathbb{R}^3} |\nabla u|^2 \leq C,$$

for $t \in [0, T^*)$. □

Lemma 4.4 *Under the conditions of Theorem 2.3.2 and (4.1), it holds that for any $T \in [0, T^*)$*

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} (|\nabla \theta|^2 + \rho |\dot{u}|^2) + \int_0^T \int_{\mathbb{R}^3} (\rho |\dot{\theta}|^2 + |\nabla \dot{u}|^2) \leq C. \quad (4.34)$$

Proof. Using the similar arguments as (3.36), we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \int_{\mathbb{R}^3} (\mu |\nabla \dot{u}|^2 + (\mu + \lambda) |\operatorname{div} \dot{u}|^2) \\ &= \int_{\mathbb{R}^3} (P_t \operatorname{div} \dot{u} + u \otimes \nabla P : \nabla \dot{u}) + \mu \int_{\mathbb{R}^3} (\operatorname{div} (\Delta u \otimes u) - \Delta (u \cdot \nabla u)) \cdot \dot{u} \\ &+ (\mu + \lambda) \int_{\mathbb{R}^3} (\operatorname{div} (\nabla \operatorname{div} u \otimes u) - \nabla \operatorname{div} (u \cdot \nabla u)) \cdot \dot{u} = \sum_{i=1}^3 IV_i. \end{aligned} \quad (4.35)$$

For IV_1 , using (2.3.1)₃ and integration by parts, we have

$$\begin{aligned}
IV_1 &= \int_{\mathbb{R}^3} \left((\rho\theta)_t \operatorname{div} \dot{u} - P(\nabla u)^t : \nabla \dot{u} - \rho\theta u \cdot \nabla \operatorname{div} \dot{u} \right) \\
&= \int_{\mathbb{R}^3} \left((\rho\theta)_t \operatorname{div} \dot{u} + \operatorname{div}(\rho\theta u) \operatorname{div} \dot{u} - P(\nabla u)^t : \nabla \dot{u} \right) \\
&= \int_{\mathbb{R}^3} \left(\rho \dot{\theta} \operatorname{div} \dot{u} - P(\nabla u)^t : \nabla \dot{u} \right) \\
&\lesssim \|\sqrt{\rho}\|_{L^\infty} \|\sqrt{\rho} \dot{\theta}\|_{L^2} \|\operatorname{div} \dot{u}\|_{L^2} + \|P\|_{L^4} \|\nabla u\|_{L^4} \|\nabla \dot{u}\|_{L^2}.
\end{aligned} \tag{4.36}$$

For IV_2 and IV_3 , by (3.38) and (3.39), we have

$$IV_2 \lesssim \|\nabla \dot{u}\|_{L^2} \|\nabla u\|_{L^4}^2, \tag{4.37}$$

and

$$IV_3 \lesssim \|\nabla \dot{u}\|_{L^2} \|\nabla u\|_{L^4}^2. \tag{4.38}$$

Substituting (4.36), (4.37) and (4.38) into (4.35), and using Cauchy inequality and (4.1), we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \int_{\mathbb{R}^3} (\mu |\nabla \dot{u}|^2 + (\mu + \lambda) |\operatorname{div} \dot{u}|^2) \leq \frac{\mu}{2} \|\nabla \dot{u}\|_{L^2}^2 + C \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4 + C.$$

Integrating this inequality over $[0, t]$ for $t \in (0, T^*)$, we have

$$\int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 \leq C \int_0^t \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + C \int_0^t \|\nabla u\|_{L^4}^4 + C. \tag{4.39}$$

The next step is to get some estimates for θ . We rewrite (2.3.1)₃ as follows:

$$\rho \dot{\theta} + \rho \theta \operatorname{div} u = \frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\operatorname{div} u)^2 + \Delta \theta. \tag{4.40}$$

Multiplying (4.40) by $\dot{\theta}$, and integrating by parts over \mathbb{R}^3 , we have

$$\begin{aligned}
\int_{\mathbb{R}^3} \rho |\dot{\theta}|^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla \theta|^2 &= - \int_{\mathbb{R}^3} \rho \theta \operatorname{div} u \dot{\theta} + \int_{\mathbb{R}^3} \left(\frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\operatorname{div} u)^2 \right) \theta_t \\
&\quad + \int_{\mathbb{R}^3} \left(\frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\operatorname{div} u)^2 \right) u \cdot \nabla \theta + \int_{\mathbb{R}^3} \Delta \theta u \cdot \nabla \theta \\
&= \sum_{i=1}^4 V_i.
\end{aligned} \tag{4.41}$$

For V_1 , using Cauchy inequality, (4.1) and (4.17), we have

$$V_1 \leq \frac{1}{8} \int_{\mathbb{R}^3} \rho |\dot{\theta}|^2 + C. \tag{4.42}$$

For V_2 , using Hölder inequality, (4.1) and (4.17), we have

$$\begin{aligned}
V_2 &= \frac{d}{dt} \int_{\mathbb{R}^3} \left(\frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\operatorname{div} u)^2 \right) \theta + \mu \int_{\mathbb{R}^3} (\nabla u + (\nabla u)') : (\nabla u_t + (\nabla u_t)') \theta \\
&\quad + 2\lambda \int_{\mathbb{R}^3} \operatorname{div} u \operatorname{div} u_t \theta \\
&= \frac{d}{dt} \int_{\mathbb{R}^3} \left(\frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\operatorname{div} u)^2 \right) \theta + \mu \int_{\mathbb{R}^3} (\nabla u + (\nabla u)') : (\nabla \dot{u} + (\nabla \dot{u})') \theta \\
&\quad - \mu \int_{\mathbb{R}^3} (\nabla u + (\nabla u)') : (\nabla(u \cdot \nabla u) + (\nabla(u \cdot \nabla u))') \theta + 2\lambda \int_{\mathbb{R}^3} \operatorname{div} u \operatorname{div} \dot{u} \theta \\
&\quad - 2\lambda \int_{\mathbb{R}^3} \operatorname{div} u \operatorname{div}(u \cdot \nabla u) \theta \\
&\leq \frac{d}{dt} \int_{\mathbb{R}^3} \left(\frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\operatorname{div} u)^2 \right) \theta + C \|\nabla u\|_{L^2} \|\nabla \dot{u}\|_{L^2} \\
&\quad - \mu \int_{\mathbb{R}^3} (\nabla u + (\nabla u)') : (\nabla(u \cdot \nabla u) + (\nabla(u \cdot \nabla u))') \theta - 2\lambda \int_{\mathbb{R}^3} \operatorname{div} u \operatorname{div}(u \cdot \nabla u) \theta.
\end{aligned}$$

Using integration by parts, (4.17) and (4.1), we have

$$\begin{aligned}
V_2 &\leq \frac{d}{dt} \int_{\mathbb{R}^3} \left(\frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\operatorname{div} u)^2 \right) \theta + C \|\nabla \dot{u}\|_{L^2} \\
&\quad - \mu \int_{\mathbb{R}^3} (\nabla u + (\nabla u)') : (\nabla u \cdot \nabla u + (\nabla u \cdot \nabla u)') \theta \\
&\quad - \mu \int_{\mathbb{R}^3} (\nabla u + (\nabla u)') : u \cdot \nabla (\nabla u + (\nabla u)') \theta - 2\lambda \int_{\mathbb{R}^3} \operatorname{div} u (\nabla u)' : \nabla u \theta \\
&\quad - 2\lambda \int_{\mathbb{R}^3} u \cdot \nabla \operatorname{div} u \operatorname{div} u \theta \\
&\leq \frac{d}{dt} \int_{\mathbb{R}^3} \left(\frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\operatorname{div} u)^2 \right) \theta + C \|\nabla \dot{u}\|_{L^2} + C \int_{\mathbb{R}^3} |\nabla u|^3 \\
&\quad + \mu \int_{\mathbb{R}^3} \frac{|\nabla u + (\nabla u)'|^2}{2} \operatorname{div} u \theta + \mu \int_{\mathbb{R}^3} \frac{|\nabla u + (\nabla u)'|^2}{2} u \cdot \nabla \theta \\
&\quad + \lambda \int_{\mathbb{R}^3} (\operatorname{div} u)^3 \theta + \lambda \int_{\mathbb{R}^3} |\operatorname{div} u|^2 u \cdot \nabla \theta \\
&\leq \frac{d}{dt} \int_{\mathbb{R}^3} \left(\frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\operatorname{div} u)^2 \right) \theta + C \|\nabla \dot{u}\|_{L^2} + C \int_{\mathbb{R}^3} |\nabla u|^3 + C \int_{\mathbb{R}^3} |\nabla u|^2 |u| |\nabla \theta|.
\end{aligned} \tag{4.43}$$

Using Hölder inequality, Cauchy inequality, Gagliardo-Nirenberg inequality and (4.17), we have

$$\begin{aligned}
\int_{\mathbb{R}^3} |\nabla u|^2 |u| |\nabla \theta| &\lesssim \|\nabla u\|_{L^4}^2 \|u\|_{L^6} \|\nabla \theta\|_{L^3} \\
&\lesssim \|\nabla u\|_{L^4}^4 + \|\nabla u\|_{L^2}^2 \|\nabla \theta\|_{L^2} \|\nabla^2 \theta\|_{L^2} \leq C \|\nabla u\|_{L^4}^4 + C \|\nabla \theta\|_{L^2} \|\nabla^2 \theta\|_{L^2}.
\end{aligned} \tag{4.44}$$

From the standard elliptic estimates and (4.40), we have

$$\|\nabla^2 \theta\|_{L^2} \lesssim \|\rho \dot{\theta}\|_{L^2} + \|\rho \theta \operatorname{div} u\|_{L^2} + \|\nabla u\|_{L^4}^2 \leq C \|\sqrt{\rho} \dot{\theta}\|_{L^2} + C \|\nabla u\|_{L^4}^2 + C, \tag{4.45}$$

where we have used (4.1) and (4.17).

Substituting (4.45) into (4.44), and using Cauchy inequality, we have

$$\int_{\mathbb{R}^3} |\nabla u|^2 |u| |\nabla \theta| \leq \frac{1}{8} \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4 + C \|\nabla \theta\|_{L^2}^2 + C. \tag{4.46}$$

Substituting (4.46) into (4.43), we have

$$\begin{aligned} V_2 &\leq \frac{d}{dt} \int_{\mathbb{R}^3} \left(\frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda(\operatorname{div} u)^2 \right) \theta + C \|\nabla \dot{u}\|_{L^2} + C \int_{\mathbb{R}^3} |\nabla u|^3 \\ &\quad + \frac{1}{8} \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4 + C \|\nabla \theta\|_{L^2}^2 + C. \end{aligned} \quad (4.47)$$

For V_3 , using (4.46), we have

$$V_3 \lesssim \int_{\mathbb{R}^3} |\nabla u|^2 |u| |\nabla \theta| \leq \frac{1}{8} \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4 + C \|\nabla \theta\|_{L^2}^2 + C. \quad (4.48)$$

For V_4 , using Hölder inequality, Gagliardo-Nirenberg inequality, (4.17), (4.45) and Young inequality, we have

$$\begin{aligned} V_4 &\lesssim \|\Delta \theta\|_{L^2} \|u\|_{L^6} \|\nabla \theta\|_{L^3} \lesssim \|\Delta \theta\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \theta\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \theta\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\nabla \theta\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \theta\|_{L^2}^{\frac{3}{2}} \leq \frac{1}{8} \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + C \|\nabla \theta\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4 + C. \end{aligned} \quad (4.49)$$

Putting (4.42), (4.47), (4.48) and (4.49) into (4.41), and integrating the resulting inequality over $[0, t]$ for $t \in (0, T^*)$, we have

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^3} \rho |\dot{\theta}|^2 + \int_{\mathbb{R}^3} |\nabla \theta|^2 &\leq 2 \int_{\mathbb{R}^3} \left(\frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda(\operatorname{div} u)^2 \right) \theta + C \int_0^t \|\nabla \dot{u}\|_{L^2} \\ &\quad + C \int_0^t \int_{\mathbb{R}^3} |\nabla u|^3 + C \int_0^t \|\nabla u\|_{L^4}^4 + C \\ &\leq C \int_0^t \|\nabla \dot{u}\|_{L^2} + C \int_0^t \int_{\mathbb{R}^3} |\nabla u|^3 + C \int_0^t \|\nabla u\|_{L^4}^4 + C, \end{aligned} \quad (4.50)$$

where we have used (4.1), (4.4) and (4.17). Multiplying (4.50) by $2C$, and adding the resulting inequality into (4.39), we have

$$\begin{aligned} &C \int_0^t \int_{\mathbb{R}^3} \rho |\dot{\theta}|^2 + 2C \int_{\mathbb{R}^3} |\nabla \theta|^2 + \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 \\ &\leq 2C^2 \int_0^t \|\nabla \dot{u}\|_{L^2} + 2C^2 \int_0^t \int_{\mathbb{R}^3} |\nabla u|^3 + 2C^2 \int_0^t \|\nabla u\|_{L^4}^4 + C \int_0^t \|\nabla u\|_{L^4}^4 + C. \end{aligned}$$

This together with Cauchy inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} (|\nabla \theta|^2 + \rho |\dot{u}|^2) + \int_0^t \int_{\mathbb{R}^3} (\rho |\dot{\theta}|^2 + |\nabla \dot{u}|^2) \lesssim \int_0^t \int_{\mathbb{R}^3} |\nabla u|^3 + \int_0^t \|\nabla u\|_{L^4}^4 + 1 \\ &\lesssim \int_0^t \int_{\mathbb{R}^3} |\operatorname{curl} u|^3 + \int_0^t \int_{\mathbb{R}^3} |G|^3 + \int_0^t \|\operatorname{curl} u\|_{L^4}^4 + \int_0^t \|G\|_{L^4}^4 + 1 \\ &\lesssim \int_0^t \|\operatorname{curl} u\|_{L^2}^{\frac{3}{2}} \|\nabla \operatorname{curl} u\|_{L^2}^{\frac{3}{2}} + \int_0^t \|G\|_{L^2}^{\frac{3}{2}} \|\nabla G\|_{L^2}^{\frac{3}{2}} + \int_0^t \|\operatorname{curl} u\|_{L^2} \|\nabla \operatorname{curl} u\|_{L^2}^3 \\ &\quad + \int_0^t \|G\|_{L^2} \|\nabla G\|_{L^2}^3 + 1, \end{aligned} \quad (4.51)$$

where we have used (4.26) and Gagliardo-Nirenberg inequality. By (4.51), (4.1), (4.4), (4.17), (4.24), (4.27) and (4.28), we have

$$\int_{\mathbb{R}^3} (|\nabla \theta|^2 + \rho |\dot{u}|^2) + \int_0^t \int_{\mathbb{R}^3} (\rho |\dot{\theta}|^2 + |\nabla \dot{u}|^2) \lesssim \int_0^t \|\sqrt{\rho} u_t\|_{L^2}^3 + 1. \quad (4.52)$$

From (4.27), we have

$$\|\sqrt{\rho}u_t\|_{L^2} \leq \|\sqrt{\rho}\dot{u}\|_{L^2} + \|\sqrt{\rho}u \cdot \nabla u\|_{L^2} \leq \|\sqrt{\rho}\dot{u}\|_{L^2} + \epsilon\|\sqrt{\rho}u_t\|_{L^2} + C_\epsilon\|\nabla u\|_{L^2} + C.$$

Taking $\epsilon = \frac{1}{2}$, using (4.17), we have

$$\|\sqrt{\rho}u_t\|_{L^2} \lesssim \|\sqrt{\rho}\dot{u}\|_{L^2} + 1. \quad (4.53)$$

Substituting (4.53) into (4.52), and using Cauchy inequality and (4.17), we have

$$\int_{\mathbb{R}^3} (|\nabla\theta|^2 + \rho|\dot{u}|^2) + \int_0^t \int_{\mathbb{R}^3} (\rho|\dot{\theta}|^2 + |\nabla\dot{u}|^2) \lesssim \int_0^t (\|\sqrt{\rho}u_t\|_{L^2} \|\sqrt{\rho}\dot{u}\|_{L^2}^2) + 1. \quad (4.54)$$

Since $\|\sqrt{\rho}u_t\|_{L^2}$ is bounded in L^1 -norm over $(0, t)$ (see (4.17)), we use (4.54) and Gronwall inequality to get (4.34). \square

Corollary 4.5 *Under the conditions of Theorem 2.3.2 and (4.1), it holds that for any $T \in [0, T^*)$*

$$\sup_{0 \leq t \leq T} (\|\nabla G\|_{L^2} + \|\nabla \operatorname{curl} u\|_{L^2} + \|\nabla u\|_{L^6} + \|u\|_{L^\infty}) + \int_0^T \int_{\mathbb{R}^3} (\|\operatorname{div} u\|_{L^\infty}^2 + |\nabla^2 \theta|^2) \leq C. \quad (4.55)$$

Proof. It follows from (4.21) and (4.22), we have

$$\|\nabla G\|_{L^2} \lesssim \|\rho\dot{u}\|_{L^2} \leq C, \quad (4.56)$$

and

$$\begin{aligned} \int_0^T \|\operatorname{div} u\|_{L^\infty}^2 &\lesssim \int_0^T \|G\|_{L^\infty}^2 + 1 \lesssim \int_0^T \|G\|_{L^6}^2 + \int_0^T \|\nabla G\|_{L^6}^2 + 1 \\ &\lesssim \int_0^T \|\nabla G\|_{L^2}^2 + \int_0^T \|\rho\dot{u}\|_{L^6}^2 + 1 \lesssim \int_0^T \|\nabla\dot{u}\|_{L^2}^2 + 1 \leq C, \end{aligned} \quad (4.57)$$

and

$$\|\nabla \operatorname{curl} u\|_{L^2} \lesssim \|\rho\dot{u}\|_{L^2} \leq C, \quad (4.58)$$

where we have used (4.1), (4.34) and Sobolev inequality.

By (4.26), we have

$$\begin{aligned} \|\nabla u\|_{L^6} &\lesssim \|\operatorname{div} u\|_{L^6} + \|\operatorname{curl} u\|_{L^6} \lesssim \|G\|_{L^6} + \|\operatorname{curl} u\|_{L^6} + 1 \\ &\lesssim \|\nabla G\|_{L^2} + \|\nabla \operatorname{curl} u\|_{L^2} + 1 \leq C, \end{aligned} \quad (4.59)$$

where we have used (4.1), (4.4), Sobolev inequality, (4.56) and (4.58).

By (4.17), (4.59) and Sobolev inequality, we have

$$\|u\|_{L^\infty} \lesssim \|u\|_{L^6} + \|\nabla u\|_{L^6} \lesssim \|\nabla u\|_{L^2} + \|\nabla u\|_{L^6} \leq C.$$

Using (4.45), (4.34), the interpolation inequality, (4.17) and (4.59), we get

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^3} |\nabla^2 \theta|^2 &\lesssim \int_0^T \int_{\mathbb{R}^3} \rho|\dot{\theta}|^2 + \int_0^T \|\nabla u\|_{L^4}^4 + 1 \\ &\lesssim \int_0^T \|\nabla u\|_{L^2} \|\nabla u\|_{L^6}^3 + 1 \leq C. \end{aligned}$$

\square

Lemma 4.6 *Under the conditions of Theorem 2.3.2 and (4.1), it holds that for any $T \in [0, T^*)$*

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \rho |\theta_t|^2 + \int_0^T \int_{\mathbb{R}^3} |\nabla \theta_t|^2 \leq C. \quad (4.60)$$

Proof. Differentiating (2.3.1)₃ with respect to t , multiplying it by θ_t , and using integration by parts, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |\theta_t|^2 + \int_{\mathbb{R}^3} |\nabla \theta_t|^2 \\ & \leq - \int_{\mathbb{R}^3} \rho_t \left(\frac{\theta_t}{2} + u \cdot \nabla \theta + \theta \operatorname{div} u \right) \theta_t - \int_{\mathbb{R}^3} \rho (u_t \cdot \nabla \theta + u \cdot \nabla \theta_t + \theta_t \operatorname{div} u) \theta_t - \int_{\mathbb{R}^3} \rho \theta \operatorname{div} u_t \theta_t \\ & \quad + \mu \int_{\mathbb{R}^3} (\nabla u + (\nabla u)') : (\nabla u_t + (\nabla u_t)') \theta_t + 2\lambda \int_{\mathbb{R}^3} \operatorname{div} u \operatorname{div} u_t \theta_t = \sum_{i=1}^5 V I_i. \end{aligned} \quad (4.61)$$

For $V I_1$, we have

$$\begin{aligned} V I_1 &= \int_{\mathbb{R}^3} \operatorname{div}(\rho u) \left(\frac{\theta_t}{2} + u \cdot \nabla \theta + \theta \operatorname{div} u \right) \theta_t \\ &= - \int_{\mathbb{R}^3} \rho u \cdot \nabla \theta_t \left(\frac{\theta_t}{2} + u \cdot \nabla \theta + \theta \operatorname{div} u \right) - \int_{\mathbb{R}^3} \rho u \cdot \frac{\nabla \theta_t}{2} \theta_t \\ & \quad - \int_{\mathbb{R}^3} \rho u \cdot (\nabla u \cdot \nabla \theta + u \cdot \nabla \nabla \theta) \theta_t - \int_{\mathbb{R}^3} \rho u \cdot (\nabla \theta \operatorname{div} u + \theta \nabla \operatorname{div} u) \theta_t \\ &= \sum_{i=1}^4 V I_{1,i}. \end{aligned} \quad (4.62)$$

For $V I_{1,1}$, we have

$$\begin{aligned} V I_{1,1} &\leq \frac{1}{24} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C \int_{\mathbb{R}^3} \rho^2 |u|^2 |\theta_t|^2 + C \int_{\mathbb{R}^3} \rho^2 |u|^4 |\nabla \theta|^2 + C \int_{\mathbb{R}^3} \rho^2 |u|^2 |\theta|^2 |\operatorname{div} u|^2 \\ &\leq \frac{1}{24} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C \int_{\mathbb{R}^3} \rho |\theta_t|^2 + C, \end{aligned} \quad (4.63)$$

where we have used Cauchy inequality, (4.1), (4.17), (4.34) and (4.55).

For $V I_{1,2}$, using Cauchy inequality, (4.1) and (4.55) again, we have

$$V I_{1,2} \leq \frac{1}{24} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C \int_{\mathbb{R}^3} \rho |\theta_t|^2. \quad (4.64)$$

For $V I_{1,3}$, using Cauchy inequality, (4.1) and (4.55) again, along with Hölder inequality, Gagliardo-Nirenberg inequality and (4.34), we have

$$\begin{aligned} V I_{1,3} &\lesssim \int_{\mathbb{R}^3} \rho |\theta_t|^2 + \int_{\mathbb{R}^3} |\nabla u|^2 |\nabla \theta|^2 + \int_{\mathbb{R}^3} |\nabla \nabla \theta|^2 \\ &\lesssim \int_{\mathbb{R}^3} \rho |\theta_t|^2 + \|\nabla u\|_{L^6}^2 \|\nabla \theta\|_{L^3}^2 + \int_{\mathbb{R}^3} |\nabla \nabla \theta|^2 \\ &\lesssim \int_{\mathbb{R}^3} \rho |\theta_t|^2 + \|\nabla \theta\|_{L^2} \|\nabla^2 \theta\|_{L^2} + \int_{\mathbb{R}^3} |\nabla \nabla \theta|^2 \\ &\lesssim \int_{\mathbb{R}^3} \rho |\theta_t|^2 + \int_{\mathbb{R}^3} |\nabla^2 \theta|^2 + 1. \end{aligned} \quad (4.65)$$

For $VI_{1,4}$, we have

$$\begin{aligned}
VI_{1,4} &= - \int_{\mathbb{R}^3} \rho u \cdot \nabla \theta \operatorname{div} u \theta_t - \int_{\mathbb{R}^3} \rho \theta u \cdot \nabla \operatorname{div} u \theta_t \\
&\lesssim \int_{\mathbb{R}^3} \rho |\theta_t|^2 + \int_{\mathbb{R}^3} |\nabla \theta|^2 |\operatorname{div} u|^2 - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta u \cdot \nabla G \theta_t - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta u \cdot \nabla (\rho \theta) \theta_t \\
&\lesssim \int_{\mathbb{R}^3} \rho |\theta_t|^2 + \|\nabla \theta\|_{L^3}^2 \|\operatorname{div} u\|_{L^6}^2 + \int_{\mathbb{R}^3} |\nabla G|^2 - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho^2 \theta u \cdot \nabla \theta \theta_t \\
&\quad - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta^2 u \cdot \nabla \rho \theta_t \\
&\leq C \int_{\mathbb{R}^3} \rho |\theta_t|^2 + C \|\nabla^2 \theta\|_{L^2} + C + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \frac{\rho^2}{2} \theta^2 \operatorname{div} u \theta_t + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \frac{\rho^2}{2} \theta^2 u \cdot \nabla \theta_t \\
&\quad + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho^2 \theta u \cdot \nabla \theta \theta_t \leq C \int_{\mathbb{R}^3} \rho |\theta_t|^2 + C \|\nabla^2 \theta\|_{L^2}^2 + \frac{1}{24} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C,
\end{aligned} \tag{4.66}$$

where we have used Cauchy inequality, (4.1), (4.55), Hölder inequality, Gagliardo-Nirenberg inequality, (4.34), integration by parts, (4.4) and (4.17).

Substituting (4.63), (4.64), (4.65) and (4.66) into (4.62), we have

$$VI_1 \leq \frac{1}{8} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C \int_{\mathbb{R}^3} \rho |\theta_t|^2 + C \int_{\mathbb{R}^3} |\nabla^2 \theta|^2 + C. \tag{4.67}$$

For VI_2 , using Cauchy inequality, Hölder inequality, (4.1) and (4.55), we have

$$\begin{aligned}
VI_2 &= - \int_{\mathbb{R}^3} \rho u_t \cdot \nabla \theta \theta_t - \int_{\mathbb{R}^3} \rho u \cdot \nabla \theta_t \theta_t - \int_{\mathbb{R}^3} \rho |\theta_t|^2 \operatorname{div} u \\
&\leq - \int_{\mathbb{R}^3} \rho \dot{u} \cdot \nabla \theta \theta_t + \int_{\mathbb{R}^3} \rho (u \cdot \nabla) u \cdot \nabla \theta \theta_t + \frac{1}{8} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C (\|\operatorname{div} u\|_{L^\infty} + 1) \int_{\mathbb{R}^3} \rho |\theta_t|^2 \\
&\leq C \|\sqrt{\rho} \theta_t\|_{L^2} \|\dot{u}\|_{L^6} \|\nabla \theta\|_{L^3} + C \|\sqrt{\rho} \theta_t\|_{L^2} \|\nabla u\|_{L^6} \|\nabla \theta\|_{L^3} + \frac{1}{8} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 \\
&\quad + C (\|\operatorname{div} u\|_{L^\infty} + 1) \int_{\mathbb{R}^3} \rho |\theta_t|^2.
\end{aligned}$$

This together with Sobolev inequality, Gagliardo-Nirenberg inequality, (4.55) and (4.34), we have

$$\begin{aligned}
VI_2 &\leq C \|\sqrt{\rho} \theta_t\|_{L^2} (\|\nabla \dot{u}\|_{L^2} + 1) \|\nabla \theta\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \theta\|_{L^2}^{\frac{1}{2}} + \frac{1}{8} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C (\|\operatorname{div} u\|_{L^\infty} + 1) \int_{\mathbb{R}^3} \rho |\theta_t|^2 \\
&\leq \frac{1}{8} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C (\|\operatorname{div} u\|_{L^\infty} + \|\nabla \dot{u}\|_{L^2}^2 + 1) \int_{\mathbb{R}^3} \rho |\theta_t|^2 + C \|\nabla^2 \theta\|_{L^2}^2 + C.
\end{aligned} \tag{4.68}$$

For VI_3 , we have

$$\begin{aligned}
VI_3 &= - \int_{\mathbb{R}^3} \rho \theta \operatorname{div} \dot{u} \theta_t + \int_{\mathbb{R}^3} \rho \theta \operatorname{div} (u \cdot \nabla u) \theta_t \\
&\leq C \int_{\mathbb{R}^3} \rho |\theta_t|^2 + C \int_{\mathbb{R}^3} |\operatorname{div} \dot{u}|^2 + C \int_{\mathbb{R}^3} \rho |\theta| |\nabla u|^2 |\theta_t| + \int_{\mathbb{R}^3} \rho \theta \theta_t u \cdot \nabla \operatorname{div} u \\
&\leq C \int_{\mathbb{R}^3} \rho |\theta_t|^2 + C \int_{\mathbb{R}^3} |\operatorname{div} \dot{u}|^2 + C \int_{\mathbb{R}^3} |\nabla u|^4 + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta \theta_t u \cdot \nabla G \\
&\quad + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho^2 \theta \theta_t u \cdot \nabla \theta + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta^2 \theta_t u \cdot \nabla \rho \\
&\leq C \int_{\mathbb{R}^3} \rho |\theta_t|^2 + C \int_{\mathbb{R}^3} |\operatorname{div} \dot{u}|^2 + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta^2 \theta_t u \cdot \nabla \rho + C,
\end{aligned} \tag{4.69}$$

where we have used (4.1), Cauchy inequality, the interpolation inequality, (4.17), (4.34) and (4.55).

To handle the third term of the right hand side of (4.69), we use integration by parts. More precisely,

$$\begin{aligned} \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta^2 \theta_t u \cdot \nabla \rho &= - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \frac{\rho^2}{2} \theta^2 \theta_t \operatorname{div} u - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \frac{\rho^2}{2} \theta^2 u \cdot \nabla \theta_t \\ &\quad - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho^2 \theta \theta_t u \cdot \nabla \theta \\ &\leq C \int_{\mathbb{R}^3} \rho |\theta_t|^2 + \frac{1}{8} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C, \end{aligned} \quad (4.70)$$

where we have used Cauchy inequality, (4.1), (4.4), (4.17), (4.34) and (4.55).

Substituting (4.70) into (4.69), we have

$$VI_3 \leq \frac{1}{8} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C \int_{\mathbb{R}^3} \rho |\theta_t|^2 + C \int_{\mathbb{R}^3} |\operatorname{div} \dot{u}|^2 + C. \quad (4.71)$$

Similar to V_2 , for VI_4 and VI_5 , we deduce

$$\begin{aligned} VI_4 + VI_5 &\leq C \|\nabla \dot{u}\|_{L^2} \|\nabla u\|_{L^3} \|\theta_t\|_{L^6} + C \int_{\mathbb{R}^3} |\nabla u|^3 |\theta_t| + C \int_{\mathbb{R}^3} |\nabla u|^4 + \frac{1}{16} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 \\ &\leq \frac{1}{16} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C \|\nabla \dot{u}\|_{L^2} \|\theta_t\|_{L^6} + C \|\nabla u\|_{L^{\frac{18}{5}}}^3 \|\theta_t\|_{L^6} + C \\ &\leq \frac{1}{16} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C (\|\nabla \dot{u}\|_{L^2} + 1) \|\nabla \theta_t\|_{L^2} + C \\ &\leq \frac{1}{8} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 + C, \end{aligned} \quad (4.72)$$

where we have used Hölder inequality, integration by parts, Cauchy inequality, (4.17), (4.55), the interpolation inequality and Sobolev inequality.

Putting (4.67), (4.68), (4.71) and (4.72) into (4.61), we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |\theta_t|^2 + \int_{\mathbb{R}^3} |\nabla \theta_t|^2 &\leq C (\|\operatorname{div} u\|_{L^\infty} + \|\nabla \dot{u}\|_{L^2}^2 + 1) \int_{\mathbb{R}^3} \rho |\theta_t|^2 \\ &\quad + C \int_{\mathbb{R}^3} (|\nabla \dot{u}|^2 + |\nabla^2 \theta|^2) + C. \end{aligned} \quad (4.73)$$

By (4.73), (4.34), (4.55) and Gronwall inequality, we complete the proof of Lemma 4.6. \square

Corollary 4.7 *Under the conditions of Theorem 2.3.2 and (4.1), it holds that for any $T \in [0, T^*)$*

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |\nabla^2 \theta|^2 \leq C. \quad (4.74)$$

Proof. It follows from (4.45), (4.1), (4.17), (4.34), (4.55), (4.60) and the interpolation inequality that

$$\|\nabla^2 \theta\|_{L^2} \leq C \|\sqrt{\rho} \theta_t\|_{L^2} + C \|\sqrt{\rho} u \cdot \nabla \theta\|_{L^2} + C \leq C.$$

\square

Lemma 4.8 *Under the conditions of Theorem 2.3.2 and (4.1), it holds that for any $T \in [0, T^*)$*

$$\sup_{0 \leq t \leq T} (\|\nabla \rho\|_{L^l} + \|\rho_t\|_{L^l}) \leq C, \quad (4.75)$$

for $l = 2, q$.

Proof. The proof of the lemma is similar to the arguments as in [20, 36]. We omit it for brevity. \square

Corollary 4.9 *Under the conditions of Theorem 2.3.2 and (4.1), it holds that for any $T \in [0, T^*)$*

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} (\rho |u_t|^2 + |\nabla^2 u|^2) + \int_0^T (\|u_t\|_{D^1}^2 + \|(u, \theta)\|_{D^{2,q}}^2) \leq C. \quad (4.76)$$

Proof. Replacing f in (4.25) by u , and using the elliptic estimates, (4.1), (4.34), (4.55) and (4.75), we get

$$\begin{aligned} \|\nabla^2 u\|_{L^2} &\lesssim \|\nabla \operatorname{curl} u\|_{L^2} + \|\nabla \operatorname{div} u\|_{L^2} \lesssim \|\nabla G\|_{L^2} + \|\nabla P(\rho, \theta)\|_{L^2} + 1 \\ &\lesssim \|\nabla \rho\|_{L^2} + \|\nabla \theta\|_{L^2} + 1 \leq C. \end{aligned} \quad (4.77)$$

It follows from (4.1), (4.17), (4.34), (4.55) and (4.77) that

$$\int_{\mathbb{R}^3} \rho |u_t|^2 \lesssim \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \int_{\mathbb{R}^3} \rho |u \cdot \nabla u|^2 \leq C,$$

and

$$\int_0^t \int_{\mathbb{R}^3} |\nabla u_t|^2 \lesssim \int_0^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla(u \cdot \nabla u)|^2 \leq C.$$

By (2.3.1)₂, Hölder inequality, (4.1), (4.34), Sobolev inequality, (4.74) and (4.75), we get

$$\begin{aligned} \int_0^t \|\nabla^2 u\|_{L^q}^2 &\lesssim \int_0^t \|\rho \dot{u}\|_{L^q}^2 + \int_0^t \|\nabla P(\rho, \theta)\|_{L^q}^2 \lesssim \int_0^t \|\dot{u}\|_{L^6}^2 + \int_0^t \|\nabla \rho\|_{L^q}^2 + \int_0^t \|\nabla \theta\|_{L^q}^2 \\ &\lesssim \int_0^t \|\nabla \dot{u}\|_{L^2}^2 + \int_0^t \|\nabla \theta\|_{L^2}^2 + \int_0^t \|\nabla^2 \theta\|_{L^2}^2 + 1 \leq C. \end{aligned} \quad (4.78)$$

Using (2.3.1)₂, Hölder inequality, (4.1), (4.34), Sobolev inequality and (4.74) again, together with (4.17), (4.55), (4.60), (4.77) and (4.78), we get

$$\begin{aligned} \int_0^t \|\nabla^2 \theta\|_{L^q}^2 &\lesssim \int_0^t \|\rho \theta_t\|_{L^q}^2 + \int_0^t \|\rho u \cdot \nabla \theta\|_{L^q}^2 + \int_0^t \|\rho \theta \operatorname{div} u\|_{L^q}^2 + \int_0^t \| |\nabla u|^2 \|_{L^q}^2 \\ &\leq C \int_0^t \| |\nabla u|^2 \|_{L^q}^2 + C \leq C \int_0^t \|\nabla u\|_{L^\infty}^2 \|\nabla u\|_{L^q}^2 + C \\ &\leq C \int_0^t \|\nabla^2 u\|_{L^q}^2 + C \leq C. \end{aligned}$$

\square

By (4.17), (4.34), (4.60), (4.76), (4.74) and (4.75), we get (4.2) and (4.3). Thus, the proof of Theorem 2.3.2 is complete. \square

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References

- [1] J. T. Beale, T. Kato, A. Majda. *Remarks on the breakdown of smooth solutions for the 3-D Euler equation*. Comm. Math. Phys., 94(1984), 61-66.
- [2] D. Bresch, B. Desjardins. *On the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids*. J. Math. Pures Appl., 87(2007), 57-90.
- [3] J. Carlson, A. Jaffe, A. Wiles. *The Millennium Prize Problems*. The American Mathematical Society, Providence, RI, 2006.
- [4] Y. Cho, H. J. Choe, H. Kim. *Unique solvability of the initial boundary value problems for compressible viscous fluids*. J. Math. Pures Appl. 83(2004), 243-275.
- [5] Y. Cho, H. Kim. *Existence results for viscous polytropic fluids with vacuum*. J. Differential Equations, 228(2006), 377-411.
- [6] Y. Cho, H. Kim. *On classical solutions of the compressible Navier-Stokes equations with non-negative initial densities*. Manuscripta Math., 120(2006), 91-129.
- [7] H.J. Choe, B.J. Jin. *Regularity of weak solutions of the compressible Navier-Stokes equations*. J. Korean Math. Soc., 40 (2003), 1031-1050.
- [8] H.J. Choe, H. Kim. *Global existence of the radially symmetric solutions of the Navier-Stokes equations for the isentropic compressible fluids*. Math. Methods Appl. Sci., 28(2005), 1-28.
- [9] S.J. Ding, H.Y. Wen, C.J. Zhu. *Global classical large solutions to 1D compressible Navier-Stokes equations with density-dependent viscosity and vacuum*. J. Differential Equations, 251(2011), 1696-1725.
- [10] J.S. Fan, S. Jiang. *Blow-Up criteria for the navier-stokes equations of compressible fluids*. J.Hyper. Diff. Eqs., 5(2008), 167-185.
- [11] J.S. Fan, S. Jiang, G. Ni. *Uniform boundedness of the radially symmetric solutions of the Navier-Stokes equations for isentropic compressible fluids*. Osaka J. Math., 46(2009), 863-876.
- [12] J.S. Fan, S. Jiang, Y.B. Ou. *A blow-up criterion for compressible viscous heat-conductive flows*. Ann. I. H. Poincar-AN, 27(2010), 337-350.
- [13] D.Y. Fang, R.Z. Zi, T. Zhang. *A blow-up criterion for two dimensional compressible viscous heat-conductive flows*. arXiv:1107.4663v1 [math.AP] 23 Jul 2011.
- [14] E. Feireisl, *Dynamics of Viscous Compressible Fluids*, Oxford Univ. Press, Oxford, 2004.

- [15] E. Feireisl, A. Novotný, H. Petzeltová. *On the existence of globally defined weak solutions to the Navier-Stokes equations*. J. Math. Fluid Mech., 3(2001), 358-392.
- [16] D. Hoff. *Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data*. J. Differential Equations, 120(1995), 215-254.
- [17] D. Hoff. *Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids*. Arch. Rational Mech. Anal., 139(1997), 303-354.
- [18] T. Huang, C.Y. Wang, H.Y. Wen. *Strong solutions of the compressible nematic liquid crystal flow*. J. Differential Equations (2011), doi:10.1016/j.jde.2011.07.036.
- [19] T. Huang, C.Y. Wang, H.Y. Wen. *Blow up criterion for compressible nematic liquid crystal flows in dimension three*, Arch. Rational Mech. Anal., 2011, to appear.
- [20] X.D. Huang, J. Li, Z.P. Xin. *Serrin type criterion for the three-dimensional viscous compressible flows*. SIAM J. Math. Anal., 43(2011), 1872-1886.
- [21] X.D. Huang, J. Li, Z.P. Xin. *Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations*. arXiv:1004.4749.
- [22] X.D. Huang, J. Li, Z.P. Xin. *Blowup criterion for the compressible flows with vacuum states*. Commun. Math. Phys., 301(2011), 23-35.
- [23] N. Itaya. *On the Cauchy problem for the system of fundamental equations describing the movement of compressible viscous fluid*. Kodai Math. Sem. Rep., 23(1971), 60-120.
- [24] S. Jiang, P. Zhang. *On spherically symmetric solutions of the compressible isentropic Navier-Stokes equations*. Comm. Math. Phys., 215(2001), 559-581.
- [25] S. Jiang. *Global spherically symmetric solutions to the equations of a viscous polytropic ideal gas in an exterior domain*. Comm. Math. Phys., 178(1996), 339-374.
- [26] B. Kawohl. *Global existence of large solutions to initial boundary value problems for a viscous, heat-conducting, one-dimensional real gas*. J. Differential Equations, 58(1985), 76-103.
- [27] A.V. Kazhikhov, V.V. Shelukhi. *Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas*. Prikl. Mat. Meh., 41(1977), 282-291.
- [28] P. L. Lions, *Mathematical topics in fluid mechanics. Vol. 2. Compressible models*. Oxford Lecture Series in Mathematics and its Applications, 10. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998. xiv+348 pp.
- [29] T. Luo, Z.P. Xin, T. Yang. *Interface behavior of compressible Navier-Stokes equations with vacuum*. SIAM J. Math. Anal., 31(2000), 1175-1191.
- [30] A. Matsumura, T. Nishida. *The initial value problem for the equations of motion of viscous and heat-conductive gases*. J. Math. Kyoto Univ., 20(1980), 67-104.
- [31] A. Matsumura, T. Nishida. *The initial boundary value problems for the equations of motion of compressible and heat-conductive fluids*. Comm. Math. Phys., 89(1983), 445-464.

- [32] G. Ponce. *Remarks on a paper: "Remarks on the breakdown of smooth solutions for the 3-D Euler equations"*. Comm. Math. Phys., 98(1985), 349-353.
- [33] O. Rozanova. *Blow-up of smooth highly decreasing at infinity solutions to the compressible Navier-Stokes equation*. J. Differential Equations, 245(2008), 1762-1774.
- [34] R. Salvi, I. Straškraba. *Global existence for viscous compressible fluids and their behavior as $t \rightarrow \infty$* . J. Fac. Sci. Univ. Tokyo, Sect. IA Math., 40(1993), 17-51.
- [35] J. Serrin. *On the interior regularity of weak solutions of the Navier-Stokes equations*. Arch. Rational Mech. Anal., 9(1962), 187-195.
- [36] Y.Z. Sun, C. Wang, Z.F. Zhang. *A Beale-Kato-Majda blow-up criterion for the 3-D compressible Navier-Stokes equations*. J. Math. Pures Appl., 95(2011), 36-47.
- [37] Y.Z. Sun, C. Wang, Z.F. Zhang. *A Beale-Kato-Majda criterion for three dimensional compressible viscous heat-conductive Flows*. Arch. Rational Mech. Anal., Digital Object Identifier (DOI) 10.1007/s00205-011-0407-1.
- [38] A. Tani. *On the first initial-boundary value problem of compressible viscous fluid motion*. Publ. Res. Inst. Math. Sci. Kyoto Univ., 13(1977), 193-253.
- [39] H.Y. Wen, L. Yao, C.J. Zhu. *A blow-up criterion of strong solution to a 3D viscous liquid-gas two-phase flow model with vacuum*. J. Math. Pures Appl. (2011), doi:10.1016/j.matpur.2011.09.005.
- [40] H.Y. Wen, C.J. Zhu. *Global classical large solutions to Navier-Stokes equations for viscous compressible and heat conducting fluids with vacuum*. arXiv:1103.1421v1 [math.AP] 8 Mar 2011, 1-38.
- [41] H.Y. Wen, C.J. Zhu. *Global symmetric classical and strong solutions of the full compressible Navier-Stokes equations with vacuum and large initial data*. arXiv:1109.5328v1 [math.AP] 25 Sep 2011.
- [42] Z.P. Xin. *Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density*. Comm. Pure Appl. Math., 51(1998), 229-240.